

ON SOME GEOMETRICAL NONLINEAR
THEORIES FOR THE PLASTIC BENDING,
STRETCHING AND BUCKLING
OF SANDWICH PLATES

By

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KEY TO SYMBOLS

All the symbols in this work are defined when introduced; however, for the convenience of the reader, the following is a list of the symbols used with a brief explanation of their meaning. Every effort has been made to use conventional notations of mechanics. Because tensors have been used in the formulation of the equations, the tensor conventions of reference [18]^{*} have been adopted; for example, Latin indices are used to denote space tensors (three-dimensional), while Greek indices denote subtensors. Also, in the following list, a tensor is written as either a covariant or contravariant tensor, but both forms may appear in the text. In addition, because different theories are developed, some quantities may take on different values in different sections; for these quantities, the explanation will correspond to the case where the symbol is first introduced. However, to remind the reader that in a particular section the quantity may have been redefined, a double asterisk appears just prior to the explanation.

^{*}Numbers in brackets designate references in the Bibliography.

a	limit of integration, equal to $'h + \frac{\bar{h}}{2}$
a^i	stress resultant defined by Eq. (3.31)
o_v^A, o_s^A	portions of boundary surface upon which displacements and stresses are prescribed, respectively
T^A, B^A, E^A	top, bottom and edge surfaces of plate, respectively
A^{i3}	** tensor defined by Eq. (3.19)
$A^{\alpha i \nu j}$	tensor defined by Eq. (5.102)
dA	differential area element
b	limit of integration, equal to $\frac{\bar{h}}{2}$
b^i	stress resultant defined by Eq. (3.32)
B^{i3}	** tensor defined by Eq. (3.19)
$B^{\alpha i \nu j}$	tensor defined by Eq. (5.103)
δB^i	variation of b^i , see Eq. (3.117)
c	limit of integration, equal to $-\frac{\bar{h}}{2}$
$c^i, {}_n c^i, {}_n c_\star^i$	stress resultants defined by Eqs. (3.34), (3.33), (3.49), respectively
$\delta C^i, \delta {}_n C^i, \delta {}_n C_\star^i$	variation of stress resultants, see Eqs. (3.119), (3.118), (3.144), respectively
d	limit of integration, equal to $-''h - \frac{\bar{h}}{2}$
d_{ij}	coefficients, see Sect. 6.5
D	rigidity parameter, see Eq. (6.86)
D^{i3}	** tensor defined by Eq. (3.25)
δe^{i3}	** variation of E^{i3} , see Eqs. (3.120) - (3.122)
E, E_s, E_t	Young's, secant, and tangent moduli, respectively
E^{i3}	** tensor defined by Eq. (3.25)
E^{ijkl}	tensor defined by Eq. (3.103)
o_f^i	acceleration components

$o F^i$	body force components
g	core rigidity ratio, equal to $\frac{\bar{G}_x}{\bar{G}_y}$
g_{ij}	metric tensor
G	shear modulus
G_s, G_t	secant and tangent shear moduli, respectively
G_x, G_y	shear moduli in a_1 and a_2 directions, respectively
h	layer thickness
$\hat{h}, H$	parameters, see Eqs. (6.25) and (6.83)
I_1, I_2, I_3	strain invariants, see Eqs. (3.77)
J_1, J_2	stress invariants, see Eqs. (3.83)
k, k_1	material constants corresponding to the bulk modulus for large and small strain, respectively
$\delta k^{\alpha\beta}$	variation of $K^{\alpha\beta}$, see Eq. (3.114)
K	material parameter that depends upon the amount of plastic strain
K_1, K_p	elastic and plastic buckling coefficients, respectively
$K^{\alpha\beta}$	stress resultant defined by Eq. (3.28)
$K^{\alpha\beta\gamma\delta}$	tensor defined by Eq. (5.60)
$o l^i$	stress resultant defined by Eq. (3.49)
L_x, L_y	length of plate in a_1 and a_2 directions, respectively
m	parameter, equal to $\frac{'h'E}{''h''E}$
$\delta m^{\alpha\beta}$	variation of $M^{\alpha\beta}$, see Eq. (3.114)
$m^i, M^{\alpha\beta}$	stress resultants defined by Eqs. (3.31) and (3.28), respectively
M^{ijkl}	tensor defined by Eq. (3.163)
n	integer in Eq. (6.78)
o^n_j	components of external unit normal vector of boundary surface

$\delta n^{\alpha\beta}, \delta n^{33}$	variations of $N^{\alpha\beta}$ and N^{33} , respectively, see Eqs. (3.114) and (3.116)
$N^{\alpha\beta}, N^{33}$	stress resultants defined by Eqs. (3.28) and (3.30), respectively
$p^i, n^p{}^i, n^{p*}{}^i$	stress resultants defined by Eqs. (3.34), (3.33), (3.49), respectively
P_o	parameter, see Eq. (6.83)
$\delta p^i, \delta n^p{}^i, \delta n^{p*}{}^i$	variations of $p^i, n^p{}^i, n^{p*}{}^i$, respectively, see Eqs. (3.119), (3.118), (3.144)
q	integer in Eq. (6.78)
δq^α	variation of Q^α , see Eq. (3.115)
Q^α	stress resultant defined by Eq. (3.29)
r	core shear stiffness parameter, see Eq. (6.83)
R_2	external resultant force in θ_2 direction
$R^{\alpha i \gamma j}$	tensor defined by Eq. (5.101)
$\delta_o s^i$	variation of s_o^i
δs^{ij}	variation of stress tensor s^{ij}
S	mean normal stress, equal to $\frac{1}{3} S_k^k$
S_y	uniaxial yield stress
S_o	quantity defined by Eq. (3.91) and corresponds to the octahedral shear stress of classical theory
s_o^i	stress resultant defined by Eq. (3.49)
s_*^i	stress vector
s^{ij}	stress tensor
δt^α	variation of T^α , see Eq. (3.115)
T^α	stress resultant defined by Eq. (3.29)
T^{ij}	stress deviator
u_i	displacement of middle surface
δu_i	variation of u_i

v_i	displacement components
δV_i	variation of v_i
V_o	volume of undeformed body
dV	differential volume element
w^i	stress resultant defined by Eq. (3.32)
δw^i	variation of w^i , see Eq. (3.117)
x^i	fixed rectangular Cartesian coordinates
Y_1, Y_2, Y_3	amplitude of the middle surface displacement function, see Eq. (6.78)
Z_{ij}	parameters defined by Eqs. (6.83)
α	core thickness-length ratio, equal to $\frac{\bar{h}}{L_x}$
β	wave-pattern aspect ratio, $\frac{L_y}{nL_x}$
β_{ij}	coefficients, see Sect. 6.5
v_y	uniaxial strain at yield point
v_o	quantity defined by Eq. (3.91) and corresponds to the octahedral shear strain of classical theory
γ_{ij}	** strain tensor
$o\gamma_{ij}, 1\gamma_{ij}, 2\gamma_{ij}$	** portions of strain tensor as defined by Eq. (3.68)
$\delta\Gamma_{ij}$	** variation of strain tensor γ_{ij}
$\delta_o\Gamma_{ij}, \delta_1\Gamma_{ij}, \delta_2\Gamma_{ij}$	** portions of strain tensor variation as defined by Eq. (3.151)
δ	variational symbol
δ_j^i	Kronecker delta
ϵ	parameter defined by Eq. (6.83)
ζ	ratio of moduli, equal to $\frac{E'}{E}$
θ^i	geodesic normal coordinates
$\delta_o t^i$	stress resultant defined by (3.145)

μ	Poisson's ratio
μ_v, μ_p	Poisson's ratio in the transition and plastic regions, respectively
ξ	stress ratio of face layers, equal to $\frac{\sigma'_{22}}{\sigma_{22}}$
ξ^{ijkl}	tensor defined by Eq. (3.104)
ρ_0	mass density of the undeformed body
ρ_1, ρ_2	parameters defined by Eqs. (6.83)
Σ^*	strain energy density of the undeformed body
φ	quantity defined by Eq. (5.104)
Φ	the metric form
ψ_i	displacement function, equal to $v_i(\theta^\alpha, 0)_3$
$\delta\psi_i$	variation of ψ_i
ω	constant defined by Eqs. (6.59) and (6.60)
$\frac{S_0}{\gamma_0}$	material parameter for large strain which corresponds to G_s of classical theory
$\frac{\delta S_0}{\delta \gamma_0}$	material parameter for large strain which corresponds to G_t of classical theory
$[ij, k]$	Christoffel symbol of the first kind
$\left\{ \begin{smallmatrix} i \\ j \ k \end{smallmatrix} \right\}$	Christoffel symbol of the second kind
Subscripts	
$i, j, k, \dots$	Latin indices that take on the values 1, 2 and 3
$\alpha, \beta, \gamma, \dots$	Greek indices that take on the values 1 and 2
$()_{cr}$	quantity associated with the critical buckling condition
$() _i$	covariant derivative with respect to the θ^i coordinates
$(),_i$	partial derivative with respect to the θ^i coordinates

Superscripts

$i, j, k, \dots$	Latin indices that take on the values 1, 2 and 3
$\alpha, \beta, \gamma, \dots$	Greek indices that take on the values 1 and 2
$(\cdot)$	quantity associated with the stable equilibrium position of buckling
$(\sim)$	prescribed (given) quantity
$'()$	upper layer quantity
$(\bar{})$	middle layer quantity
$''()$	lower layer quantity

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On the basis of the modified Hellinger-Reissner variational theorem of nonlinear elasticity, a general nonlinear theory is developed for sandwich plates that are stressed in the plastic region. The plasticity equations correspond to the stress-strain equations of Hencky, extended to large deformations. As a result, the material is taken to be isotropic, and the loading is assumed to be both proportional and active. The general theory is then extended to the buckling of sandwich plates by applying the bifurcation theory.

In addition, by omitting certain quadratic terms from Green's strain tensor, a simplified large deflection theory is developed for the plastic bending, stretching and buckling of sandwich plates. Furthermore, the theory is then extended to those sandwich plates which have thin outer layers; in this case, the Kirchhoff-Love hypothesis is applied. Finally, a numerical solution is obtained for the plastic

buckling of a simply supported sandwich plate with very thin outer layers and a soft core under a uniaxial compressive load.

1. INTRODUCTION

By way of introduction, it probably should be mentioned first that many man-made structures consist of flexural members, that is, plates, shells and thin bars. For these structures to behave in a predictable manner, the ability to analyze the flexural members is of vital importance. In general, flexural members fail in one of two ways, either they become overstressed or they deform excessively, in particular, the buckling phenomenon. Often, two separate analyses are required to investigate both possibilities. Well-established branches of mechanics have been developed which deal with such analyses, for example, strength of material, theory of elasticity, theory of plasticity, theory of plates, theory of shells, theory of elastic stability, etc.

In this work, one small class of flexural members will be considered, namely, a three-layered plate which is commonly called a sandwich plate. Usually, the primary purpose of a sandwich plate is to obtain a relatively rigid plate that weighs less than a solid (one material) plate of the same rigidity; however, sandwich plates have certainly been designed for other purposes.

The sandwich plate has found numerous applications in the aircraft and aerospace industries; as a result, considerable attention and work has been devoted to this type of plate in the last twenty years.

Because of their geometry, it is natural to develop a theory for sandwich plates by extending the theory of ordinary plates. Unfortunately, however, there are many theories for ordinary plates; for example, there are theories which depend upon the plate geometry (thick or thin), and the amount of deformation (small or large deflections), and the state of stress (elastic or plastic). Even more numerous theories can be developed for sandwich plates because of its structure; for example, consider the outer two layers which may be thick or thin, or different thicknesses, or different materials, etc.

Because of this multitude of theories, there comes a time when it is desirable to develop a general theory which encompasses the various specialized theories. However, by their nature, the equations in a general theory are usually too complicated to be solved, and, therefore, recourse must be taken to the simpler theories for solutions. One of the advantages of reducing a general theory to a special theory is that one is made aware of the factors being neglected. In contradistinction, when the special theory is developed directly, one is usually aware of only those factors being considered; note the difference. Because of the variety of methods available, a general theory can be developed by different techniques, for example, from one of the variational principles or directly from the three-dimensional equations of nonlinear elasticity. Now, the fundamental equations of the nonlinear theory of elasticity consist of the following: the equations of motion, the boundary conditions (both stress and displacement), the strain-displacement equations, and the stress-strain equations; these equations are the ingredients of a general plate theory.

Recently, a general theory for sandwich plates was given by Ebcioglu [1], in which Hamilton's principle was employed.

Because of its importance to this work, it is desirable to discuss some of the salient characteristics of Ebcioglu's work. The equations were derived with reference to the undeformed and stress free state of the plate, called the reference state; in this frame of reference (material or Lagrangian), there are two types of stress tensors, the Lagrangian stress tensor and the Kirchhoff stress tensor. The Kirchhoff stress tensor has the advantage of being a symmetric tensor and was used by Ebcioglu. From geometrical considerations, the most general strain-displacement relations are the Green's strain tensor (material coordinates) and the Almansi's strain tensor (spatial or Eulerian coordinates), both of which are nonlinear; Ebcioglu used Green's strain tensor. From a physical point of view, Ebcioglu took very general stress-strain equations which include thermal strain and are applicable to elastic anisotropic materials; the equations allow for the possibility of a nonlinear material.

From the preceding paragraph, it becomes apparent that the general theory can take different forms depending upon the type of stress tensor applied and/or the frame of reference used. However, in this work, an alternate form of the general theory will not be developed; instead, the theory will be developed from a different variational principle, namely, the modified Hellinger-Reissner theorem. The modified Hellinger-Reissner theorem has recently been employed by Habip [2] to develop a general theory for ordinary plates, and, later, Huang [3] applied the theory to a sandwich shell.

It should be mentioned that for a general theory there are no uniqueness theorems which state that the solution is the only possible solution; this is as it should be because it is well known that there are different deformations possible for a given environment, commonly known as the buckling phenomenon. Therefore, a special formulation is required to predict the onset of buckling. To the author's knowledge, the general theory for sandwich plates has not yet been extended to the buckling problem; the buckling equations will be developed in this work.

Although the stress-strain equations employed by Ebcioğlu may be nonlinear, it is well known that for most structural materials (i.e., steel, aluminum, etc.) a nonlinear stress-strain equation implies that the material has been stressed into the plastic range. Because the theories of plasticity and elasticity are not the same, different general theories should be developed which are applicable to both types of plate problems. In this work, we are interested in developing a general theory which is based on the equations of plasticity.

In the field of plasticity, there are many theories, usually classified as the incremental (or flow) theories and the deformation (or total strain) theories. The book by Hill [4] deals almost exclusively with the incremental theories (in particular, the Reuss equations) because these theories describe the actual behavior of materials better than the deformation theories. However, the deformation theories are simpler mathematically and, therefore, are often used, particularly Hencky's equations. It is well known that Hencky's equations can be derived as a special case of Reuss' equations, namely, when the principal

axes of stress do not change and the stress ratios remain constant, see Smith and Sidebottom [5]; this is a practical condition that occurs quite often in plates. In Hill's [4] comments on Hencky's equations, he mentions that the equations have not been extended to the case of large strains (e.g., Green's strain tensor); however, as we shall see, the foundations have been established for large strain theory.

The modified Hellinger-Reissner theorem assumes the existence of a strain energy density from which an explicit form for the stress-strain equations is obtained. However, a strain energy density implies that the material is elastic, not plastic; but, according to Novozhilov [6], Kachanov has proved that for an active deformation an elastic-plastic body is indistinguishable from an ideally elastic body, both with the same stress-strain curve. Therefore, the concept of a strain-energy density can be employed to a restrictive class of plasticity problems. Furthermore, for large strains (Green's strain tensor), Novozhilov [6] has shown that equations corresponding to Hencky's equations can be obtained in terms of a strain energy density. The procedure of Novozhilov will be used in this work; and, furthermore, explicit expressions for the strain energy density and, hence, Hencky's equations will be derived.

The theory for the plastic buckling of plates is largely a result of experience gained in the study of the plastic buckling of columns. As is well known, two different philosophies had prevailed for many years concerning the plastic buckling of columns, which were based on the assumption of whether or not strain reversal occurs during buckling. The two different approaches were unified by Shanley [7]

in 1947. As discussed by Gerard [8], von Karman concluded from Shanley's work that the stability limit should be redefined for plastic buckling as "the smallest value of the axial load at which bifurcation of the equilibrium positions can occur, regardless of whether or not the transition to the bent position requires an increase of the axial load." Therefore, the bifurcation theory for buckling will be employed in this work; a formal statement of the theory is given in Sect. 3.4. It is important to note that in the bifurcation theory the loading is active.

Based on the above discussion, a general theory for sandwich plates which includes buckling is developed. The theory is based on stress-strain equations of the Hencky type, extended for large strains. The theory is developed in Sects. 2 and 3 of this work. In addition, some specialized formulations are considered.

In bending problems, the lateral displacements are usually much larger than the displacement components in the plane of the plate. On physical grounds, it has been argued^{*} that often certain quadratic terms associated with the inplane displacements may be omitted from the strain tensor. In Sect. 4, a modified theory is developed in which only those quadratic terms of the lateral displacement are retained in the strain tensor.

Because most sandwich plates are made with thin outer layers, special consideration is given to this type of structure. As mentioned by Fung [9], probably the most important discovery in plate theory was

^{*} See Novozhilov [6].

Kirchhoff's first assumption (stated in Sect. 5) which reduced the equations to a form where practical solutions could readily be obtained. Kirchhoff's assumption along with Love's assumption are basic hypotheses in current thin plate theory. In Sect. 5, a theory for sandwich plates with thin facings is developed which utilizes the Kirchhoff-Love hypothesis along with the modified strain tensor described in the preceding paragraph.

Finally, in Sect. 6, a sample problem is worked out in detail and compared with the work of previous investigators.

2. THE VARIATIONAL PRINCIPLE

It is well known that in the nonlinear theory of elasticity the basic equations can be obtained from variational principles, in particular, Reissner's theorem. Because of the early work of Hellinger^{*} [10], the variational principle is often referred to as the Hellinger-Reissner theorem, see Truesdell and Toupin [12]. Rather than work with the theorem as stated by Reissner [13], an alternate form developed by Washizu [14] is more convenient for our purposes, referred to as the modified Hellinger-Reissner theorem. Habip [2] has used the modified Hellinger-Reissner theorem to develop a general nonlinear plate theory; in this work, the procedure used will be quite similar to that given by Habip. The variational principle as given by Habip is

"the modified Hellinger-Reissner theorem asserts that the variational principle

$$\begin{aligned} \int_V \rho_0 ({}_0f^i - {}_0F^i) \delta v_i dV &= \delta \left[\int_V (S^{ij} \gamma_{ij} - \Sigma^*) dV \right. \\ &- \int_V \frac{1}{2} S^{ij} (v_{i|j} + v_{j|i} + v^r{}_{|i} v_{r|j}) dV \\ &\left. + \int_{A_s} \tilde{S}_*^i v_i dA + \int_{A_v} S_*^i (v_i - \tilde{v}_i) dA \right] \end{aligned} \quad (2.1)$$

^{*} According to Naghdi [11], Hellinger's work did not include the boundary integrals.

where γ_{ij} , S^{ij} , v_i and $S_\star^i$ are varied independently, is equivalent to Cauchy's first law in ${}_oV$, to the stress boundary condition on the part ${}_oA_s$ of the boundary, to the displacement boundary condition on the remaining part ${}_oA_v$, and to the stress-strain and strain-displacement relations in ${}_oV$, when the symmetries of γ_{ij} and S^{ij} are both used."

All of the various quantities in the above theorem are referenced to a body that is undeformed and stress free, known as the reference state; for example, the symbol ${}_oV$ denotes the volume of the undeformed body, i.e., the volume in the reference state. With the understanding, hereafter, that all of the quantities are with respect to the reference state, the meaning of the various symbols used in the statement of the theorem is as follows:

${}_oA_v$ is the portion of the boundary of the body upon which the displacement vectors are prescribed (given);

${}_oA_s$ is the remaining part of the boundary upon which the stress vectors are prescribed;

${}_of^i$ are components of acceleration;

${}_oF^i$ are components of body force per unit mass;

S^{ij} are contravariant components of the stress tensor;

$S_\star^i$ are contravariant components of the stress vector;

v^i (v_i) are contravariant (covariant) components of the displacement vector;

${}_oV$ is the volume of the body;

γ_{ij} are covariant components of the strain tensor;

δ is the variational symbol used in calculus of variations;

ρ_o is the mass density of the body;

$\Sigma^\star$ is the strain energy density per unit volume;

- | is the covariant derivative symbol used in tensor calculus;
 ~ is the tilde symbol which is used to denote prescribed quantities.

When the indicated variation is performed, Eq. (2.1) can be put into the form

$$\begin{aligned}
 & \int_V \{ [S^{ij} (\delta_i^r + v^r|_i)]|_j + \rho_0 ({}_0F^r - {}_0f^r) \} \delta v_r dV \\
 & + \int_{A_s} [\tilde{S}_*^i - {}_0m_j S^{jr} (\delta_r^i + v^i|_r)] \delta v_i dA \\
 & + \int_{A_r} (v_i - \tilde{v}_i) \delta S_*^i dA \\
 & + \int_V [\gamma_{ij} - \frac{1}{2} (v_i|_j + v_j|_i + v^r|_i v_r|_j)] \delta S^{ij} dV \\
 & + \int_V [S^{ij} - \frac{1}{2} (\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}})] \delta \gamma_{ij} dV = 0
 \end{aligned} \tag{2.2}$$

where Gauss' theorem

$$\int_V (S^{ij} \delta v_j)|_i dV = \int_A S^{ij} \delta v_j {}_0n_i dA \tag{2.3}$$

has been employed and where the strain energy density has been assumed to be a function of the strain tensor, i.e.,

$$\Sigma^* = \Sigma^*(\gamma_{ij}).$$

In Eq. (2.2), δ_i^r is the Kronecker delta, and ${}_0n_j$ is the external unit normal vector of the boundary surface. Then from the generalization and extension of the basic lemma of calculus of variations (hereafter,

referred to as the fundamental theorem of calculus of variations), as indicated by Weinstock [15], Eq. (2.2) yields the fundamental equations of the nonlinear theory of elasticity in terms of the reference state as follows:

the equations of motion in ${}_0V$ (Cauchy's first law),

$$[S^{jk}(\delta_k^i + v^i|_k)]|_j + \rho_0({}_0F^i - {}_0f^i) = 0; \quad (2.4)$$

the boundary conditions on ${}_0A_s$,

$$\tilde{S}_*^i - {}_0m_j S^{jk}(\delta_k^i + v^i|_k) = 0; \quad (2.5)$$

the boundary conditions on ${}_0A_v$,

$$v_i - \tilde{v}_i = 0; \quad (2.6)$$

the strain-displacement relations in ${}_0V$,

$$\gamma_{ij} = \frac{1}{2} (v_i|_j + v_j|_i + v^r|_i v_r|_j); \quad (2.7)$$

the stress-strain relations (in terms of the strain energy density) in ${}_0V$,

$$S^{ij} = \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right). \quad (2.8)$$

These equations are geometrically nonlinear; and depending upon Σ^* , the equations can also be physically nonlinear. Later, in Sect. 3.3.4,

it will be shown that by a proper choice of Σ^* the equations are applicable to a certain class of plasticity problems. Perhaps it should be mentioned that in the literature γ_{ij} is known as Green's strain tensor and S_{ij} is known as Kirchhoff's stress tensor.

Examination of the modified Hellinger-Reissner theorem shows that by certain modifications, which are practically self-evident, equations corresponding to alternate theories can be obtained. For example, in the classical theory of elasticity, the strain displacement equation is

$$\gamma_{ij} = \frac{1}{2} (v_{i|j} + v_{j|i});$$

it is obvious that this expression can be obtained from the theorem by omitting the quadratic term

$$v^r|_i v_r|_j$$

from the third integral in Eq. (2.1). Thus, Eq. (2.1) would read

$$\begin{aligned} \int_V \rho_0 ({}_0f^i - {}_0F^i) \delta v_i dV = & \delta \left[\int_V (S^{ij} \gamma_{ij} - \Sigma^*) dV \right. \\ & - \int_V \frac{1}{2} S^{ij} (v_{i|j} + v_{j|i}) dV + \int_{A_s} \tilde{S}_*^i v_i dA \\ & \left. + \int_{A_v} S_*^i (v_i - \tilde{v}_i) dA \right] \end{aligned}$$

which after carrying out the indicated variation becomes

$$\begin{aligned}
& \int_V [S^{ij}|_j + \rho_0 ({}_0F^i - {}_0f^i)] \delta v_i dV \\
& + \int_{A_s} (\tilde{S}_*^i - S^{ij} {}_0m_j) \delta v_i dA + \int_{A_v} (v_i - \tilde{v}_i) \delta S_*^i dA \\
& + \int_V [\gamma_{ij} - \frac{1}{2} (v_{i|j} + v_{j|i})] \delta S^{ij} dV \\
& + \int_V [S^{ij} - \frac{1}{2} (\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}})] \delta \gamma_{ij} dV = 0.
\end{aligned}$$

The application of the fundamental theorem of calculus of variation, indeed, yields the classical equations of elasticity, provided of course, that S^{ij} is now interpreted as the Eulerian stress tensor and Σ^* is chosen to yield Hooke's law.

In the study of plates, theories have been developed which are based on both the general nonlinear equations and the classical equations. In addition, intermediate theories have been developed, for example, the work of Ebcioğlu [16] in which a strain tensor was employed as follows:

$$\gamma_{ij} = \frac{1}{2} (v_{i|j} + v_{j|i} + v_{3,i} v_{3,j})$$

where the comma denotes the partial derivative; a more restrictive form of the above strain tensor is used in the well-known von Karman theory. Because virtually all plate theories are based on simplifying assumptions, it is sometimes difficult to establish a consistent

set of equations, particularly when the equations are developed by the direct method^{*} which is often employed. In general, this difficulty is overcome by using the modified Hellinger-Reissner theorem because by introducing the simplifying assumptions in a systematic manner the theorem automatically produces a consistent set of equations.

^{*} In the direct method the basic equations are obtained by considering the stress resultants acting on a plate element and then applying the equations of equilibrium.

3. GENERAL NONLINEAR THEORY OF A SANDWICH PLATE

3.1. Description of the Sandwich Plate

In this work, a sandwich plate is a three-layer, laminar structure of constant thickness. The layers may be dissimilar materials of different thicknesses; however, the thickness of each individual layer is constant. In addition, each layer is intimately fixed in relation to the other so that the sandwich plate, as a whole, is a continuous medium; thus, the displacements are continuous throughout, including across the interfaces. As a result of the application of Gauss' theorem to obtain Eq. (2.2), the plate must be a regular region;^{*} therefore, there are no cavities or holes through the plate. Also, it is desirable to require that the edge surface is a portion of a right cylindrical surface. Naturally, the terminology "plate" implies a flat structure (as distinguished from a shell which has a curved surface), the thickness of which is small as compared with its other dimensions.

To properly describe the sandwich plate, a coordinate system^{**} must be chosen; naturally, curvilinear coordinates should be chosen to utilize the general form of the variational principle. However, to

^{*} See Kellogg [17].

^{**} All coordinate systems will be taken as right-handed.

take advantage of the geometry of a plate, the curvilinear coordinates can be specialized without introducing a loss in generality to the theory; these specialized coordinates and the rectangular Cartesian coordinates to which they are referred are shown in Fig. 1.

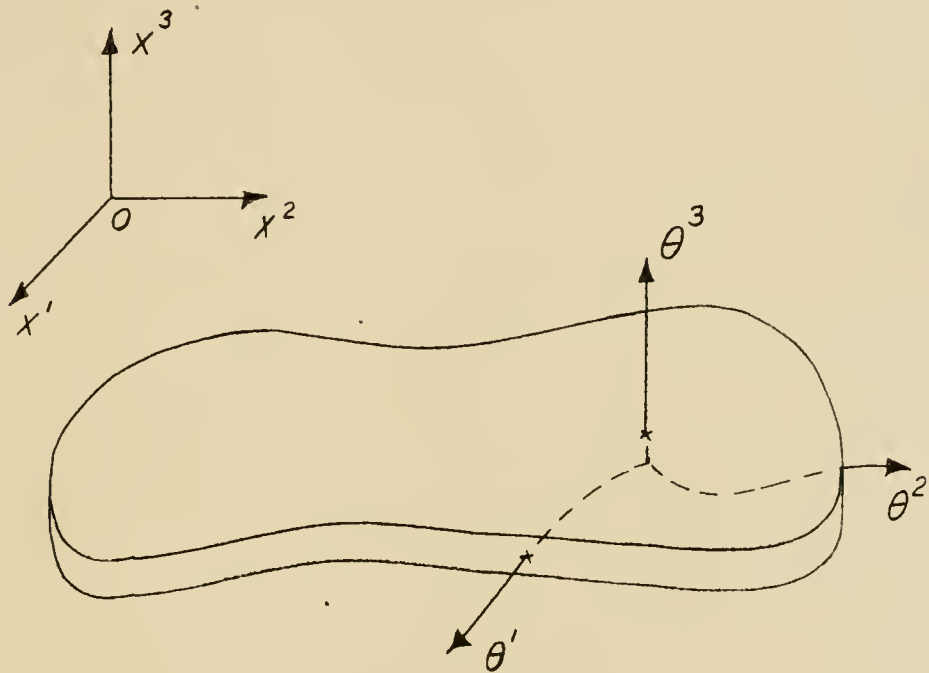


Fig. 1. Plate coordinates

Here, the θ^3 coordinate curve is rectilinear and orthogonal to the θ^1 and θ^2 coordinate curves; furthermore, the functional relationships between the θ^i and x^i coordinates are

$$\theta^\alpha = \theta^\alpha(x^\beta), \quad \theta^3 = x^3 + \text{const.},$$

and

$$x^\alpha = x^\alpha(\theta^\beta), \quad x^3 = \theta^3 + \text{const.},$$

where the convention is now introduced that the Greek indices ($\alpha, \beta, \gamma, \dots$) take only the values 1, 2. Note that all planes parallel to the upper and lower faces of the plate are described by equations of the type

$$\theta^3 = \text{const.}$$

By introducing a translation of coordinates, any plane parallel to the faces can be written as

$$\bar{\theta}^3 = \theta^3 - \text{const.} = 0. \quad (3.1)$$

In plate theory, the usual procedure is to write the equations in terms of coordinates such that the middle plane is described by an equation like Eq. (3.1). Therefore, it is convenient to choose the coordinate system for the sandwich plate as shown in Fig. 2 where the middle planes of the upper, middle, and lower layers are, respectively, described by the equations

$$' \theta^3 = 0, \quad \bar{\theta}^3 = 0, \quad '' \theta^3 = 0; \quad (3.2)$$

and are related by the equations

$$\begin{aligned} ' \theta^3 &= \bar{\theta}^3 - \frac{1}{2} (\bar{h} + 'h), \\ '' \theta^3 &= \bar{\theta}^3 + \frac{1}{2} (\bar{h} + ''h), \end{aligned} \quad (3.3)$$

where $'h$, $\bar{h}$, $''h$ are the thicknesses of the upper, middle, and lower layers, respectively. Hereafter, the notation of a prime, a bar, and

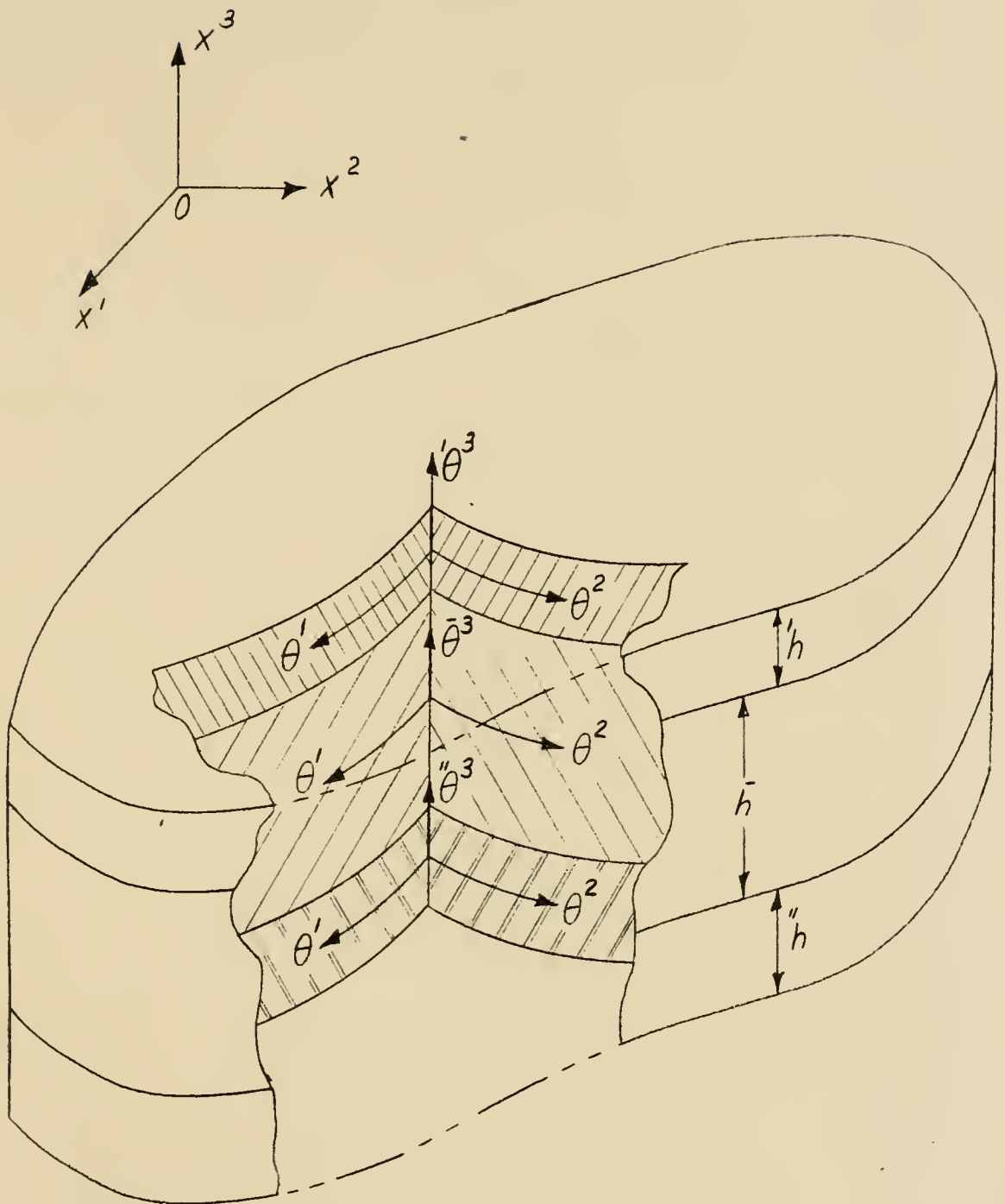


Fig. 2. Coordinates for sandwich plate

a double prime will be used to distinguish quantities associated with the upper, middle, and lower layers, respectively.

For the above coordinate system, the metric form

$$\Phi = g_{ij} d\theta^i d\theta^j$$

becomes

$$\Phi = g_{\alpha\beta} d\theta^\alpha d\theta^\beta + (d\theta^3)^2$$

where, from the definition of the metric tensor,

$$g_{\alpha\beta} = \sum_{\gamma=1}^2 \frac{\partial x^\gamma}{\partial \theta^\alpha} \frac{\partial x^\gamma}{\partial \theta^\beta},$$

$$g_{\alpha 3} = 0, \quad g_{33} = 1.0. \quad (3.4)$$

As a result it is easy to show that all the Christoffel symbols with the index 3 become zero, i.e.,

$$[i3, j] = [3i, j] = [ij, 3] = 0,$$

$$\left\{ \begin{smallmatrix} i \\ j \end{smallmatrix} 3 \right\} = \left\{ \begin{smallmatrix} i \\ 3 \end{smallmatrix} j \right\} = \left\{ \begin{smallmatrix} 3 \\ i \end{smallmatrix} j \right\} = 0; \quad (3.5)$$

also

$$g^{\alpha\beta} g_{\alpha\gamma} = \delta_\gamma^\beta, \quad g^{\alpha 3} = 0, \quad g^{33} = 1.0. \quad (3.6)$$

Because of the above characteristics,

$$\begin{aligned} v^\alpha &= g^{\alpha\beta} v_\beta, & v_\alpha &= g_{\alpha\beta} v^\beta, & v^3 &= v_3, \\ v^i/{}_3 &= v^i{}_{,3}, & v_i/{}_3 &= v_{i,3}, & (3.7) \\ v^3/{}_i &= v_{3,i}, & v_3/{}_i &= v_{3,i}, \end{aligned}$$

where the comma denotes the ordinary partial derivative with respect to the x^i coordinates.

The coordinate system is the Euclidean space case of the geodesic normal coordinate system discussed by Synge and Schild [18]; as a result, when a space tensor, say S^{ij} , is split up into the components $S^{\alpha\beta}$, $S^{\alpha 3}$, S^{33} , the components are known as subtensors,* subvectors and subinvariants, respectively.

3.2. Displacement Functions

Throughout the region of the plate, the displacement is assumed to be a sufficiently well-behaved function that can be expanded in a Taylor series, at least to the first two terms. For our purposes, it is desirable to expand v^i in terms of θ^3 about the plane $\theta^3 = 0$, i.e.,

$$v^i(\theta^j) = v^i(\theta^\alpha, 0) + \theta^3 [v^i(\theta^\alpha, 0)_{,3}].$$

* Some authors use the terminology surface tensors, e.g., see Green and Zerna [19].

By introducing the notation

$$u^i(\theta^\alpha) = v^i(\theta^\alpha, 0),$$

$$\psi^i(\theta^\alpha) = v^i(\theta^\alpha, 0)_{,3},$$

the displacement can be written as

$$v^i(\theta^j) = u^i(\theta^\alpha) + \theta^3 \psi^i(\theta^\alpha). \quad (3.8)$$

The function u^i represents the displacement of a particle in the middle surface. As mentioned by Novozhilov [6], the function ψ^i characterizes the direction of a fiber in the strained state which was initially normal to the middle plane. Note that fibers which were initially normal to the middle plane are not necessarily normal to the middle surface after deformation, as assumed in many plate theories.

By extending the above to a sandwich plate, the displacement of particles in each of the three layers can be written as

$$\begin{aligned} v^i(\theta^j) &= u^i(\theta^\alpha) + \theta^3 \psi^i(\theta^\alpha), \\ \bar{v}^i(\bar{\theta}^j) &= \bar{u}^i(\theta^\alpha) + \bar{\theta}^3 \bar{\psi}^i(\theta^\alpha), \\ {}''v^i({}''\theta^j) &= {}''u^i(\theta^\alpha) + {}''\theta^3 {}''\psi^i(\theta^\alpha). \end{aligned} \quad (3.9)$$

By use of Eqs. (3.3), the equations can be rewritten in terms of the coordinates of the middle layer, i.e.,

$$\begin{aligned}
{}^i v(\bar{\theta}^j) &= {}^i u(\theta^\alpha) + [\bar{\theta}^3 - \frac{1}{2}(\bar{h} + {}^i h)] {}^i \psi(\theta^\alpha), \\
\bar{v}^i(\bar{\theta}^j) &= \bar{u}^i(\theta^\alpha) + \bar{\theta}^3 \bar{\psi}^i(\theta^\alpha), \\
{}^{ii} v(\bar{\theta}^j) &= {}^{ii} u(\theta^\alpha) + [\bar{\theta}^3 + \frac{1}{2}(\bar{h} + {}^{ii} h)] {}^{ii} \psi(\theta^\alpha).
\end{aligned} \tag{3.10}$$

These equations can be related further by making use of the fact that the displacements must be equal at the interfaces (continuity conditions)

$$\begin{aligned}
{}^i v &= \bar{v}^i, \quad \text{at} \quad \bar{\theta}^3 = \frac{\bar{h}}{2}, \\
{}^{ii} v &= \bar{v}^i, \quad \text{at} \quad \bar{\theta}^3 = -\frac{\bar{h}}{2}.
\end{aligned} \tag{3.11}$$

Substitution of Eqs. (3.10) into (3.11) gives

$$\begin{aligned}
{}^i u &= \bar{u}^i + \frac{1}{2} [\bar{h} \bar{\psi}^i + {}^i h {}^i \psi^i] \\
{}^{ii} u &= \bar{u}^i - \frac{1}{2} [\bar{h} \bar{\psi}^i + {}^{ii} h {}^{ii} \psi^i],
\end{aligned} \tag{3.12}$$

or

$$\begin{aligned}
\bar{u}^i &= \frac{1}{2} [{}^i u + {}^{ii} u - \frac{1}{2} ({}^i h {}^i \psi^i - {}^{ii} h {}^{ii} \psi^i)] \\
\bar{\psi}^i &= \frac{1}{\bar{h}} [{}^i u - {}^{ii} u - \frac{1}{2} ({}^i h {}^i \psi^i + {}^{ii} h {}^{ii} \psi^i)].
\end{aligned} \tag{3.13}$$

Therefore, the displacements can be written as either

$$\begin{aligned}
{}^{\prime}\bar{v}^i &= \bar{u}^i + \frac{\bar{h}}{2} (\bar{\psi}^i - {}^{\prime}\psi^i) + \bar{\theta}^3 {}^{\prime}\psi^i \\
\bar{v}^i &= \bar{u}^i + \bar{\theta}^3 \bar{\psi}^i \\
{}^{\prime\prime}\bar{v}^i &= \bar{u}^i - \frac{\bar{h}}{2} (\bar{\psi}^i - {}^{\prime\prime}\psi^i) + \bar{\theta}^3 {}^{\prime\prime}\psi^i,
\end{aligned} \tag{3.14}$$

or

$$\begin{aligned}
{}^{\prime}\bar{v}^i &= {}^{\prime}u^i + [\bar{\theta}^3 - \frac{1}{2}(\bar{h} + {}^{\prime}h)] {}^{\prime}\psi^i \\
\bar{v}^i &= \frac{1}{2} [{}^{\prime}u^i + {}^{\prime\prime}u^i - \frac{1}{2}({}^{\prime}h {}^{\prime}\psi^i - {}^{\prime\prime}h {}^{\prime\prime}\psi^i)] \\
&\quad + \frac{\bar{\theta}^3}{\bar{h}} [{}^{\prime}u^i - {}^{\prime\prime}u^i - \frac{1}{2}({}^{\prime}h {}^{\prime}\psi^i + {}^{\prime\prime}h {}^{\prime\prime}\psi^i)] \\
{}^{\prime\prime}\bar{v}^i &= {}^{\prime\prime}u^i + [\bar{\theta}^3 + \frac{1}{2}(\bar{h} + {}^{\prime\prime}h)] {}^{\prime\prime}\psi^i.
\end{aligned} \tag{3.15}$$

In applying the variational principle, either Eqs. (3.14) or Eqs. (3.15) can be substituted into Eq. (2.2). If the former are used, the resulting expressions will be in terms of the quantities

$$\bar{u}^i, \bar{\psi}^i, {}^{\prime}\psi^i, {}^{\prime\prime}\psi^i;$$

or, if Eqs. (3.15) are employed, the results will be in terms of

$${}^{\prime}u^i, {}^{\prime\prime}u^i, {}^{\prime}\psi^i, {}^{\prime\prime}\psi^i.$$

In either case, there will be twelve unknown displacement functions to be determined. Regardless of which set of variables one chooses

to work with, the resulting equations can be reformulated in terms of the other variables through Eqs. (3.12) or (3.13); this will be demonstrated later for a simple case. Because Eqs. (3.14) are more compact than Eq. (3.15), the equations will be formulated in terms of the variables

$$\bar{a}^i, \bar{\psi}^i, {}^i\psi, {}^i\psi.$$

3.3. Fundamental Equations

3.3.1. Equations of motion

Because the variational principle leads to a rather lengthy equation, i.e., Eq. (2.2), which is awkward to handle in its entirety, it is desirable to split up the equation if possible. Examination of Eqs. (2.2) and (2.4) shows that the equation of motion is a direct consequence of the first integral of Eq. (2.2). Therefore, the equation of motion can be obtained from the equation

$$\int_V \{ [S^{ij} (\delta_i^r + v^r |_{\delta_i})] |_{\delta_j} + \rho_0 (F^r - f^r) \} \delta v_r dV = 0. \quad (3.16)$$

When written in terms of the subtensors, as discussed in Sect. 3.1, Eq. (3.16) becomes

$$\begin{aligned}
& \int_V \left\{ \left[S^{\beta\gamma} (\delta_\gamma^\alpha + v^\alpha|_\gamma) + S^{\beta 3} v^\alpha_{,3} \right] |_\beta + \rho_0 ({}_0F^\alpha - {}_0f^\alpha) \right. \\
& \quad \left. + \left[S^{\gamma 3} (\delta_\gamma^\alpha + v^\alpha|_\gamma) + S^{33} v^\alpha_{,3} \right]_{,3} \right\} \delta v_\alpha \\
& \quad + \left\{ \left[S^{\beta\gamma} v_{3,\gamma} + S^{\beta 3} (1 + v_{3,3}) \right] |_\beta + \rho_0 ({}_0F^3 - {}_0f^3) \right. \\
& \quad \left. + \left[S^{\gamma 3} v_{3,\gamma} + S^{33} (1 + v_{3,3}) \right]_{,3} \right\} \delta v_3 \Big] dV = 0.
\end{aligned} \tag{3.17}$$

For future use, it is convenient to rewrite Eq. (3.17) as follows:

$$\begin{aligned}
& \int_V \left\{ \left[A^{\alpha\beta} |_\beta + B^{\alpha 3}_{,3} + \rho_0 ({}_0F^\alpha - {}_0f^\alpha) \right] \delta v_\alpha \right. \\
& \quad \left. + \left[A^{3\beta} |_\beta + B^{33}_{,3} + \rho_0 ({}_0F^3 - {}_0f^3) \right] \delta v_3 \right\} dV = 0
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
A^{i\beta} &= S^{\beta\gamma} (\delta_\gamma^i + v^i|_\gamma) + S^{\beta 3} (\delta_3^i + v^i_{,3}) \\
B^{i3} &= S^{\gamma 3} (\delta_\gamma^i + v^i|_\gamma) + S^{33} (\delta_3^i + v^i_{,3}).
\end{aligned} \tag{3.19}$$

The total volume ${}_0V$ of the sandwich plate consists of the sum of the upper layer volume $'V$, the middle layer volume $\bar{V}$, and the lower layer volume $''V$. Therefore, for a sandwich plate, Eq. (3.18) becomes

$$\begin{aligned}
& \int_{\dot{V}} \left\{ \left[\dot{A}^{\alpha\beta} /_{\beta} + \dots \right] \delta \dot{v}_{\alpha} + \left[\dot{A}^{3\beta} /_{\beta} + \dots \right] \delta \dot{v}_3 \right\} d\dot{V} \\
& + \int_{\bar{V}} \left\{ \left[\bar{A}^{\alpha\beta} /_{\beta} + \dots \right] \delta \bar{v}_{\alpha} + \left[\bar{A}^{3\beta} /_{\beta} + \dots \right] \delta \bar{v}_3 \right\} d\bar{V} \\
& + \int_{\ddot{V}} \left\{ \left[\ddot{A}^{\alpha\beta} /_{\beta} + \dots \right] \delta \ddot{v}_{\alpha} + \left[\ddot{A}^{3\beta} /_{\beta} + \dots \right] \delta \ddot{v}_3 \right\} d\ddot{V} = 0.
\end{aligned} \tag{3.20}$$

Note that

$$d\dot{V} = d\dot{\theta}^3 dA, \quad d\bar{V} = d\bar{\theta}^3 dA, \quad d\ddot{V} = d\ddot{\theta}^3 dA \tag{3.21}$$

where

$$dA = \sqrt{\text{DET. } g_{\alpha\beta}} d\theta^{\alpha}.$$

Then, with Eqs. (3.21), the integrals of Eq. (3.20) can be combined to obtain the equation

$$\begin{aligned}
& \int_A \left\{ \int_b^a \left[(\dot{A}^{\alpha\beta} /_{\beta} + \dots) \delta \dot{v}_{\alpha} + (\dot{A}^{3\beta} /_{\beta} + \dots) \delta \dot{v}_3 \right] \right. \\
& + \int_c^b \left[(\bar{A}^{\alpha\beta} /_{\beta} + \dots) \delta \bar{v}_{\alpha} + (\bar{A}^{3\beta} /_{\beta} + \dots) \delta \bar{v}_3 \right] \\
& \left. + \int_d^c \left[(\ddot{A}^{\alpha\beta} /_{\beta} + \dots) \delta \ddot{v}_{\alpha} + (\ddot{A}^{3\beta} /_{\beta} + \dots) \delta \ddot{v}_3 \right] \right\} d\bar{\theta}^3 dA = 0
\end{aligned} \tag{3.22}$$

where the limits are

$$\begin{aligned}
a &= 'h + \frac{\bar{h}}{2}, & b &= \frac{\bar{h}}{2}, \\
c &= -\frac{\bar{h}}{2}, & d &= -'h - \frac{\bar{h}}{2}.
\end{aligned} \tag{3.23}$$

Expressions for the displacements were established in Sect. 3.2; thus, substitution of Eqs. (3.14) into Eqs. (3.22) gives

$$\begin{aligned}
 & \int_A \left[\left\{ \int_b^a \left['D^{\alpha\beta} /_\beta + 'E^{\alpha 3}{}_{,3} + ' \rho ('F^\alpha - 'f^\alpha) \right] \right. \right. \\
 & \quad \left. \left. + \int_c^b \left[\bar{D}^{\alpha\beta} /_\beta + \dots \right] + \int_d^c \left[''D^{\alpha\beta} /_\beta + \dots \right] \right\} \delta \bar{u}_\alpha \right. \\
 & \quad \left. + \int_b^a \frac{\bar{h}}{2} \left['D^{\alpha\beta} /_\beta + \dots \right] + \int_c^b \bar{\theta}^3 \left[\bar{D}^{\alpha\beta} /_\beta + \dots \right] \right. \\
 & \quad \left. - \int_d^c \frac{\bar{h}}{2} \left[''D^{\alpha\beta} /_\beta + \dots \right] \right\} \delta \bar{\psi}_\alpha + \int_b^a \left(\bar{\theta}^3 - \frac{\bar{h}}{2} \right) \left['D^{\alpha\beta} /_\beta + \dots \right] \delta \psi'_\alpha \\
 & \quad \left. + \int_d^c \left(\bar{\theta}^3 + \frac{\bar{h}}{2} \right) \left[''D^{\alpha\beta} /_\beta + \dots \right] \delta \psi''_\alpha \right. \\
 & \quad \left. + \left\{ \int_b^a \left['D^{3\alpha} /_\alpha + 'E^{33}{}_{,3} + ' \rho ('F^3 - 'f^3) \right] \right. \right. \\
 & \quad \left. \left. + \int_c^b \left[\bar{D}^{3\alpha} /_\alpha + \dots \right] + \int_d^c \left[''D^{3\alpha} /_\alpha + \dots \right] \right\} \delta \bar{u}_3 \right. \\
 & \quad \left. + \int_b^a \frac{\bar{h}}{2} \left['D^{3\alpha} /_\alpha + \dots \right] + \int_c^b \bar{\theta}^3 \left[\bar{D}^{3\alpha} /_\alpha + \dots \right] \right. \\
 & \quad \left. - \int_d^c \frac{\bar{h}}{2} \left[''D^{3\alpha} /_\alpha + \dots \right] \right\} \delta \bar{\psi}_3 + \int_b^a \left(\bar{\theta}^3 - \frac{\bar{h}}{2} \right) \left['D^{3\alpha} /_\alpha + \dots \right] \delta \psi'_3 \\
 & \quad \left. + \int_d^c \left(\bar{\theta}^3 + \frac{\bar{h}}{2} \right) \left[''D^{3\alpha} /_\alpha + \dots \right] \delta \psi''_3 \right] d\bar{\theta}^3 dA = 0
 \end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
 \dot{D}^{i\beta} &= \dot{S}^{\beta\gamma} \left[\delta_{\gamma}^i + \bar{u}^i /_{\gamma} + \frac{\hbar}{2} (\bar{\psi}^i /_{\gamma} - \dot{\psi}^i /_{\gamma}) \right. \\
 &\quad \left. + \bar{\theta}^3 \dot{\psi}^i /_{\gamma} \right] + \dot{S}^{\beta 3} (\delta_3^i + \dot{\psi}^i), \\
 \bar{D}^{i\beta} &= \bar{S}^{\beta\gamma} (\delta_{\gamma}^i + \bar{u}^i /_{\gamma} + \bar{\theta}^3 \bar{\psi}^i /_{\gamma}) \\
 &\quad + \bar{S}^{\beta 3} (\delta_3^i + \bar{\psi}^i), \\
 {}''D^{i\beta} &= {}''S^{\beta\gamma} \left[\delta_{\gamma}^i + \bar{u}^i /_{\gamma} - \frac{\hbar}{2} (\bar{\psi}^i /_{\gamma} - {}''\psi^i /_{\gamma}) \right. \\
 &\quad \left. + \bar{\theta}^3 {}''\psi^i /_{\gamma} \right] + {}''S^{\beta 3} (\delta_3^i + {}''\psi^i), \\
 \dot{E}^{i3} &= \dot{S}^{\gamma 3} \left[\delta_{\gamma}^i + \bar{u}^i /_{\gamma} + \frac{\hbar}{2} (\bar{\psi}^i /_{\gamma} - {}''\psi^i /_{\gamma}) \right. \\
 &\quad \left. + \bar{\theta}^3 \dot{\psi}^i /_{\gamma} \right] + \dot{S}^{33} (\delta_3^i + \dot{\psi}^i), \\
 \bar{E}^{i3} &= \bar{S}^{\gamma 3} (\delta_{\gamma}^i + \bar{u}^i /_{\gamma} + \bar{\theta}^i /_{\gamma}) \\
 &\quad + \bar{S}^{33} (\delta_3^i + \bar{\psi}^i), \\
 {}''E^{i3} &= {}''S^{\gamma 3} \left[\delta_{\gamma}^i + \bar{u}^i /_{\gamma} - \frac{\hbar}{2} (\bar{\psi}^i /_{\gamma} - {}''\psi^i /_{\gamma}) \right. \\
 &\quad \left. + \bar{\theta}^3 {}''\psi^i /_{\gamma} \right] + {}''S^{33} (\delta_3^i + {}''\psi^i).
 \end{aligned} \tag{3.25}$$

Note that

$$\begin{aligned} \int_b^a \dot{E}^{\alpha 3}{}_{,3} d\bar{\theta}^3 &= (\dot{E}^{\alpha 3})_b^a, \\ \int_b^a \dot{E}^{\alpha 3}{}_{,3} \bar{\theta}^3 d\bar{\theta}^3 &= (\bar{\theta}^3 \dot{E}^{\alpha 3})_b^a - \int_b^a \dot{E}^{\alpha 3} d\bar{\theta}^3. \end{aligned} \quad (3.26)$$

Therefore, Eq. (3.24) becomes

$$\begin{aligned} & \int_A \left\{ \left[\int_b^a [\dot{D}^{\alpha\beta}|_\beta + \dot{\rho}_\alpha (\dot{F}^\alpha - f^\alpha)] d\bar{\theta}^3 \right. \right. \\ & + \int_c^b [\bar{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 + \int_d^c [\ddot{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 \\ & + (\dot{E}^{\alpha 3})_b^a + (\bar{E}^{\alpha 3})_c^b + (\ddot{E}^{\alpha 3})_d^c \left. \right\} \delta \bar{u}_\alpha \\ & + \left\{ \int_b^a \frac{\bar{h}}{2} [\dot{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 + \int_c^b \bar{\theta}^3 [\bar{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 \right. \\ & - \int_d^c \frac{\bar{h}}{2} [\ddot{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 - \int_c^b \bar{E}^{\alpha 3} d\bar{\theta}^3 + \frac{\bar{h}}{2} (\dot{E}^{\alpha 3})_b^a \\ & + (\bar{\theta}^3 \bar{E}^{\alpha 3})_c^b - \frac{\bar{h}}{2} (\ddot{E}^{\alpha 3})_d^c \left. \right\} \delta \bar{\psi}_\alpha \\ & + \left\{ \int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) [\dot{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 - \int_b^a \dot{E}^{\alpha 3} d\bar{\theta}^3 \right. \\ & + (\bar{\theta}^3 \dot{E}^{\alpha 3})_b^a - \frac{\bar{h}}{2} (\dot{E}^{\alpha 3})_b^a \left. \right\} \delta \psi_\alpha \\ & + \left\{ \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) [\ddot{D}^{\alpha\beta}|_\beta + \dots] d\bar{\theta}^3 - \int_d^c \ddot{E}^{\alpha 3} d\bar{\theta}^3 + \right. \end{aligned}$$

$$\begin{aligned}
& + (\bar{\theta}^3 \ddot{E}^{33})_d^c + \frac{\bar{h}}{2} (\ddot{E}^{33})_d^c \} \delta'' \psi_\alpha \\
& + \left\{ \int_b^a \left[\dot{D}^{3\alpha}|_\alpha + \dot{\rho}_0 (\dot{F}^3 - \dot{f}^3) \right] d\bar{\theta}^3 \right. \\
& + \int_c^b \left[\bar{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 + \int_d^c \left[\ddot{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 \\
& + (\dot{E}^{33})_b^a + (\bar{E}^{33})_c^b + (\ddot{E}^{33})_d^c \} \delta \bar{u}_3 \\
& + \int_b^a \frac{\bar{h}}{2} \left[\dot{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 + \int_c^b \bar{\theta}^3 \left[\bar{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 \\
& - \int_d^c \frac{\bar{h}}{2} \left[\ddot{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 - \int_c^b \bar{E}^{33} d\bar{\theta}^3 \\
& + \frac{\bar{h}}{2} (\dot{E}^{33})_b^a + (\bar{\theta}^3 \bar{E}^{33})_c^b + (\ddot{E}^{33})_d^c \} \delta \bar{\psi}_3 \\
& + \left\{ \int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) \left[\dot{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 - \int_b^a \dot{E}^{33} d\bar{\theta}^3 \right. \\
& + (\bar{\theta}^3 \dot{E}^{33})_b^a - \frac{\bar{h}}{2} (\dot{E}^{33})_b^a \} \delta' \psi_3 \\
& + \left\{ \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) \left[\ddot{D}^{3\alpha}|_\alpha + \dots \right] d\bar{\theta}^3 - \int_d^c \ddot{E}^{33} d\bar{\theta}^3 \right. \\
& + (\bar{\theta}^3 \ddot{E}^{33})_d^c + \frac{\bar{h}}{2} (\ddot{E}^{33})_d^c \} \delta'' \psi_3 \Big] dA = 0.
\end{aligned}$$

Next, stress resultants are defined as follows:

$$\begin{Bmatrix} 'N^{\alpha\beta}, \bar{N}^{\alpha\beta}, ''N^{\alpha\beta} \\ 'M^{\alpha\beta}, \bar{M}^{\alpha\beta}, ''M^{\alpha\beta} \\ 'K^{\alpha\beta}, \bar{K}^{\alpha\beta}, ''K^{\alpha\beta} \end{Bmatrix} = \begin{Bmatrix} \int_b^a 'S^{\alpha\beta}, \int_c^b \bar{S}^{\alpha\beta}, \int_d^c ''S^{\alpha\beta} \end{Bmatrix} \begin{Bmatrix} 1 \\ \bar{\theta}^3 \\ (\bar{\theta}^3)^2 \end{Bmatrix} d\bar{\theta}^3; \quad (3.28)$$

$$\begin{Bmatrix} 'Q^\alpha, \bar{Q}^\alpha, ''Q^\alpha \\ 'T^\alpha, \bar{T}^\alpha, ''T^\alpha \end{Bmatrix} = \begin{Bmatrix} \int_b^a 'S^{\alpha 3}, \int_c^b \bar{S}^{\alpha 3}, \int_d^c ''S^{\alpha 3} \end{Bmatrix} \begin{Bmatrix} 1 \\ \bar{\theta}^3 \end{Bmatrix} d\bar{\theta}^3; \quad (3.29)$$

$$\{ 'N^{33}, \bar{N}^{33}, ''N^{33} \} = \left\{ \int_b^a 'S^{33}, \int_c^b \bar{S}^{33}, \int_d^c ''S^{33} \right\} d\bar{\theta}^3; \quad (3.30)$$

$$\begin{Bmatrix} 'a^i, \bar{a}^i, ''a^i \\ 'm^i, \bar{m}^i, ''m^i \end{Bmatrix} = \begin{Bmatrix} \int_b^a \rho_o 'f^i, \int_c^b \bar{\rho}_o \bar{f}^i, \int_d^c ''\rho_o ''f^i \end{Bmatrix} \begin{Bmatrix} 1 \\ \bar{\theta}^3 \end{Bmatrix} d\bar{\theta}^3; \quad (3.31)$$

$$\begin{Bmatrix} 'b^i, \bar{b}^i, ''b^i \\ 'w^i, \bar{w}^i, ''w^i \end{Bmatrix} = \begin{Bmatrix} \int_b^a \rho_o 'F^i, \int_c^b \bar{\rho}_o \bar{F}^i, \int_d^c ''\rho_o ''F^i \end{Bmatrix} \begin{Bmatrix} 1 \\ \bar{\theta}^3 \end{Bmatrix} d\bar{\theta}^3; \quad (3.32)$$

$$\begin{Bmatrix} m'p^i, m'c^i \\ m\bar{p}^i, m\bar{c}^i \\ m''p^i, m''c^i \end{Bmatrix} = \begin{Bmatrix} 'E^{i3}, \bar{\theta}^3 'E^{i3} \\ \bar{E}^{i3}, \bar{\theta}^3 \bar{E}^{i3} \\ ''E^{i3}, \bar{\theta}^3 ''E^{i3} \end{Bmatrix}, \quad AT \begin{Bmatrix} '\theta^3 \\ \bar{\theta}^3 \\ ''\theta^3 \end{Bmatrix} = (-1)^m \begin{Bmatrix} \frac{h}{2} \\ \bar{h} \\ \frac{h}{2} \end{Bmatrix}; \quad (3.33)$$

$$\begin{Bmatrix} 'p^i, 'c^i \\ \bar{p}^i, \bar{c}^i \\ ''p^i, ''c^i \end{Bmatrix} = \begin{Bmatrix} 'b^i + {}_2'p^i - {}_1'p^i, 'w^i + {}_2'c^i - {}_1'c^i \\ \bar{b}^i + {}_2\bar{p}^i - {}_1\bar{p}^i, \bar{w}^i + {}_2\bar{c}^i - {}_1\bar{c}^i \\ ''b^i + {}_2''p^i - {}_1''p^i, ''w^i + {}_2''c^i - {}_1''c^i \end{Bmatrix}. \quad (3.34)$$

By applying the fundamental theorem of calculus of variations to Eq. (3.27), and by using Eqs. (3.28) - (3.34), the equations of motion are finally obtained, and can be put in the following form:

$$\begin{aligned}
 & \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + {}''N^{\beta\gamma})(\delta_\gamma^\alpha + \bar{u}^\alpha|_\gamma) + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma})' \psi^\alpha|_\gamma \right. \\
 & + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}^\alpha|_\gamma + ({}''M^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi^\alpha|_\gamma \\
 & + \dot{Q}^\beta \psi^\alpha + \bar{Q}^\beta \bar{\psi}^\alpha + {}''Q^\beta {}''\psi^\alpha \Big\} |_\beta + \dot{p}^\alpha + \bar{p}^\alpha + {}''p^\alpha \\
 & = \dot{a}^\alpha + \bar{a}^\alpha + {}''a^\alpha ; \tag{3.35}
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] (\delta_\gamma^\alpha + \bar{u}^\alpha|_\gamma) + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma})' \psi^\alpha|_\gamma \right. \\
 & + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + {}''N^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}^\alpha|_\gamma - \frac{\bar{h}}{2} ({}''M^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi^\alpha|_\gamma \\
 & + \frac{\bar{h}}{2} (\dot{Q}^\beta \psi^\alpha - {}''Q^\beta {}''\psi^\alpha) \Big\} |_\beta - \bar{Q}^\gamma (\delta_\gamma^\alpha + \bar{u}^\alpha|_\gamma) \\
 & + (\bar{T}^\beta|_\beta - \bar{N}^{33}) \bar{\psi}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - {}''p^\alpha) + \bar{c}^\alpha \\
 & = \frac{\bar{h}}{2} (\dot{a}^\alpha - {}''a^\alpha) + \bar{m}^\alpha ; \tag{3.36}
 \end{aligned}$$

$$\begin{aligned}
& \left\{ (M^{\beta\gamma} - \frac{\bar{h}}{2} N^{\beta\gamma}) (\delta_\gamma^\alpha + \bar{u}^\alpha /_\gamma + \frac{\bar{h}}{2} \bar{\psi}^\alpha /_\gamma) - \frac{\bar{h}}{2} Q^\beta \psi^\alpha \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 N^{\beta\gamma} - \bar{h} M^{\beta\gamma} + K^{\beta\gamma} \right] \psi^\alpha /_\gamma \left. \right\} |_\beta + (T^\beta /_\beta - N^{33}) \psi^\alpha \\
& - Q^\gamma \left[\delta_\gamma^\alpha + \bar{u}^\alpha /_\gamma + \frac{\bar{h}}{2} (\bar{\psi}^\alpha /_\gamma - \psi^\alpha /_\gamma) \right] - \frac{\bar{h}}{2} p^\alpha + c^\alpha \\
& = - \frac{\bar{h}}{2} a^\alpha + m^\alpha ; \tag{3.37}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (M^{\beta\gamma} + \frac{\bar{h}}{2} N^{\beta\gamma}) (\delta_\gamma^\alpha + \bar{u}^\alpha /_\gamma - \frac{\bar{h}}{2} \bar{\psi}^\alpha /_\gamma) + \frac{\bar{h}}{2} Q^\beta \bar{\psi}^\alpha \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 N^{\beta\gamma} + \bar{h} M^{\beta\gamma} + K^{\beta\gamma} \right] \bar{\psi}^\alpha /_\gamma \left. \right\} |_\beta + (T^\beta /_\beta - N^{33}) \bar{\psi}^\alpha \\
& - Q^\gamma \left[\delta_\gamma^\alpha + \bar{u}^\alpha /_\gamma - \frac{\bar{h}}{2} (\bar{\psi}^\alpha /_\gamma - \psi^\alpha /_\gamma) \right] + \frac{\bar{h}}{2} p^\alpha + c^\alpha \\
& = \frac{\bar{h}}{2} a^\alpha + m^\alpha ; \tag{3.38}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (N^{\beta\gamma} + \bar{N}^{\beta\gamma} + N^{\beta\gamma}) \bar{u}_{3,\gamma} + (M^{\beta\gamma} - \frac{\bar{h}}{2} N^{\beta\gamma}) \psi_{3,\gamma} \right. \\
& + \left[\frac{\bar{h}}{2} (N^{\beta\gamma} - N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + (M^{\beta\gamma} + \frac{\bar{h}}{2} N^{\beta\gamma}) \psi_{3,\gamma} \\
& + Q^\beta (1 + \psi_3) + \bar{Q}^\beta (1 + \bar{\psi}_3) + Q^\beta (1 + \psi_3) \left. \right\} |_\beta \\
& + p_3 + \bar{p}_3 + p_3 = a_3 + \bar{a}_3 + a_3 \tag{3.39}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}_{3,\gamma} \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\psi}_{3,\gamma} \\
& + \frac{\bar{h}}{2} \dot{Q}^\beta (1 + \dot{\psi}_3) - \frac{\bar{h}}{2} \ddot{Q}^\beta (1 + \ddot{\psi}_3) \left. \right\} |_\beta + (\bar{T}^\beta |_\beta - \bar{N}^{33}) (1 + \bar{\psi}_3) \\
& - \bar{Q}^\beta \bar{u}_{3,\beta} + \frac{\bar{h}}{2} (\dot{p}_3 - \ddot{p}_3) + \bar{c}_3 = \frac{\bar{h}}{2} (\dot{a}_3 - \ddot{a}_3) + \bar{m}_3 ;
\end{aligned} \tag{3.40}$$

$$\begin{aligned}
& \left\{ (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) (\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) - \frac{\bar{h}}{2} \dot{Q}^\beta (1 + \dot{\psi}_3) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\psi}_{3,\gamma} \left. \right\} |_\beta + (\dot{T}^\beta |_\beta - \dot{N}^{33}) (1 + \dot{\psi}_3) \\
& - \dot{Q}^\gamma \left[\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma}) \right] - \frac{\bar{h}}{2} \dot{p}_3 + \dot{c}_3 \\
& = - \frac{\bar{h}}{2} \dot{a}_3 + \dot{m}_3 ;
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
& \left\{ (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) (\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) + \frac{\bar{h}}{2} \ddot{Q}^\beta (1 + \ddot{\psi}_3) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\psi}_{3,\gamma} \left. \right\} |_\beta + (\ddot{T}^\beta |_\beta - \ddot{N}^{33}) (1 + \ddot{\psi}_3) \\
& - \ddot{Q}^\gamma \left[\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \ddot{\psi}_{3,\gamma}) \right] + \frac{\bar{h}}{2} \ddot{p}_3 + \ddot{c}_3 \\
& = \frac{\bar{h}}{2} \ddot{a}_3 + \ddot{m}_3 .
\end{aligned} \tag{3.42}$$

3.3.2. Boundary conditions

It is obvious that the boundary conditions, i.e., Eqs. (2.5) and (2.6), are a direct result of the second and third integrals, respectively, of Eq. (2.2). Therefore, as in the preceding section, the boundary conditions can be obtained from the equations

$$\int_{\partial A_S} [\tilde{S}_*^i - m_j S^{jr} \{ \delta_r^i + v^i |_{,r} \}] \delta v_i dA = 0, \quad (3.43)$$

$$\int_{\partial A_v} (v_i - \tilde{v}_i) \delta S_*^i dA = 0. \quad (3.44)$$

The total boundary ∂A of the sandwich plate consists of the top surface ${}_T A$, bottom surface ${}_B A$, and the edge surface ${}_E A$. In addition, the edge surface consists of portions of the upper, middle, and lower layers; thus,

$$\partial A = {}_T A + {}_B A + ({}_E A + {}_E \bar{A} + {}_E A). \quad (3.45)$$

Let us first establish the stress boundary conditions which are obtained from Eq. (3.43). In terms of subtensors, Eq. (3.43) can be written as

$$\begin{aligned}
& \int_{\circ A_S} \left\{ \left[\tilde{S}_*^\alpha - {}_\circ m_\beta S^{\beta\gamma} (\delta_\gamma^\alpha + v^\alpha|_\gamma) - {}_\circ m_\beta S^{\beta 3} v^\alpha_{,3} \right. \right. \\
& \quad \left. \left. - {}_\circ m_3 S^{3\gamma} (\delta_\gamma^\alpha + v^\alpha|_\gamma) - {}_\circ m_3 S^{33} v^\alpha_{,3} \right] \delta v_\alpha \right. \\
& \quad \left. + \left[\tilde{S}_*^3 - {}_\circ m_\beta S^{\beta\gamma} v_{3,\gamma} - {}_\circ m_\beta S^{\beta 3} (1 + v_{3,3}) \right. \right. \\
& \quad \left. \left. - {}_\circ m_3 S^{3\gamma} v_{3,\gamma} - {}_\circ m_3 S^{33} (1 + v_{3,3}) \right] \delta v_3 \right\} dA = 0,
\end{aligned}$$

or

$$\begin{aligned}
& \int_{\circ A_S} \left[(\tilde{S}_*^\alpha - {}_\circ m_\beta A^{\alpha\beta} - {}_\circ m_3 B^{\alpha 3}) \delta v_\alpha \right. \\
& \quad \left. + (\tilde{S}_*^3 - {}_\circ m_\beta A^{3\beta} - {}_\circ m_3 B^{33}) \delta v_3 \right] dA = 0 \quad (3.46)
\end{aligned}$$

where Eqs. (3.19) have been employed. Because of the geometry and coordinate system, observe that:

on ${}_T A_S$,

$${}'_\circ m_\beta = 0, \quad {}'_\circ m_3 = 1.0;$$

on ${}_B A_S$,

$${}''_\circ m_\beta = 0, \quad {}''_\circ m_3 = -1.0;$$

and on ${}_E A_S$,

$${}'_\circ m_\beta = {}_\circ \bar{m}_\beta = {}''_\circ m_\beta, \quad {}'_\circ m_3 = {}_\circ \bar{m}_3 = {}''_\circ m_3 = 0.$$

Also, note that the edge integrals can be written as

$$\begin{aligned}\int_{E A_S'} (\cdots) dA &= \int_a \int_b^a (\cdots) d\bar{\theta}^3 d\omega, \\ \int_{E \bar{A}_S} (\cdots) dA &= \int_a \int_c^b (\cdots) d\bar{\theta}^3 d\omega, \\ \int_{E A_S''} (\cdots) dA &= \int_a \int_d^c (\cdots) d\bar{\theta}^3 d\omega.\end{aligned}$$

Therefore, Eq. (3.46) can now be written as

$$\begin{aligned}& \int_{A_S'} [(\tilde{S}_*^{\alpha} - B^{\alpha 3}) \delta v_{\alpha} + (\tilde{S}_*^3 - B^{33}) \delta v_3] dA \\& + \int_{A_S''} [(\tilde{S}_*^{\alpha} + B^{\alpha 3}) \delta v_{\alpha} + (\tilde{S}_*^3 + B^{33}) \delta v_3] dA \\& + \int_a \left\{ \int_b^a [(\tilde{S}_*^{\alpha} - \bar{m}_{\beta} A^{\alpha \beta}) \delta v_{\alpha} + (\tilde{S}_*^3 - \bar{m}_{\beta} A^{3 \beta}) \delta v_3] d\bar{\theta}^3 \right. \\& + \int_c^b [(\tilde{S}_*^{\alpha} - \bar{m}_{\beta} \bar{A}^{\alpha \beta}) \delta \bar{v}_{\alpha} + (\tilde{S}_*^3 - \bar{m}_{\beta} \bar{A}^{3 \beta}) \delta \bar{v}_3] d\bar{\theta}^3 \\& \left. + \int_d^c [(\tilde{S}_*^{\alpha} - \bar{m}_{\beta} A^{\alpha \beta}) \delta v_{\alpha} + (\tilde{S}_*^3 - \bar{m}_{\beta} A^{3 \beta}) \delta v_3] d\bar{\theta}^3 \right\} d\omega = 0.\end{aligned}\tag{3.47}$$

Next, by substituting the displacement functions as given by Eqs. (3.14),

Eq. (3.47) becomes

$$\begin{aligned}
& \int_{\tau A_5} \left\{ (\tilde{S}_*^\alpha - E^{\alpha 3}) \left[\delta \bar{u}_\alpha + \frac{\bar{h}}{2} \delta \bar{\psi}_\alpha + (\bar{\theta}^3 - \frac{\bar{h}}{2}) \delta \psi'_\alpha \right] \right. \\
& + (\tilde{S}_*^3 - E^{33}) \left[\delta \bar{u}_3 + \frac{\bar{h}}{2} \delta \bar{\psi}_3 + (\bar{\theta}^3 - \frac{\bar{h}}{2}) \delta \psi'_3 \right] \left. \right\} dA \\
& + \int_{\sigma A_5} \left\{ (\tilde{S}_*^\alpha + E^{\alpha 3}) \left[\delta \bar{u}_\alpha - \frac{\bar{h}}{2} \delta \bar{\psi}_\alpha + (\bar{\theta}^3 + \frac{\bar{h}}{2}) \delta \psi''_\alpha \right] \right. \\
& + (\tilde{S}_*^3 + E^{33}) \left[\delta \bar{u}_3 - \frac{\bar{h}}{2} \delta \bar{\psi}_3 + (\bar{\theta}^3 + \frac{\bar{h}}{2}) \delta \psi''_3 \right] \left. \right\} dA \\
& + \int_{\sigma \omega} \left\{ \left[\int_b^a (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) + \int_c^b (\tilde{S}_*^\alpha - \bar{m}_\beta \bar{D}^{\alpha\beta}) + \int_d^c (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) \right] \delta \bar{u}_\alpha \right. \\
& + \left[\int_b^a \frac{\bar{h}}{2} (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) + \int_c^b \bar{\theta}^3 (\tilde{S}_*^\alpha - \bar{m}_\beta \bar{D}^{\alpha\beta}) - \int_d^c \frac{\bar{h}}{2} (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) \right] \delta \bar{\psi}_\alpha \\
& + \int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) \delta \psi'_\alpha \\
& + \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) (\tilde{S}_*^\alpha - \bar{m}_\beta D^{\alpha\beta}) \delta \psi''_\alpha \\
& + \left[\int_b^a (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) + \int_c^b (\tilde{S}_*^3 - \bar{m}_\beta \bar{D}^{3\beta}) + \int_d^c (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) \right] \delta \bar{u}_3 \\
& + \left[\int_b^a \frac{\bar{h}}{2} (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) + \int_c^b \bar{\theta}^3 (\tilde{S}_*^3 - \bar{m}_\beta \bar{D}^{3\beta}) - \int_d^c \frac{\bar{h}}{2} (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) \right] \delta \bar{\psi}_3 \\
& + \int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) \delta \psi'_3 \\
& + \left. \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) (\tilde{S}_*^3 - \bar{m}_\beta D^{3\beta}) \delta \psi''_3 \right\} d\bar{\theta}^3 d\omega = 0
\end{aligned}$$

(3.48)

where Eqs. (3.25) have been employed.

Now, let us introduce the following definitions:

$$\begin{aligned} \left\{ \begin{matrix} {}_2'p_*^i, {}_2'c_*^i \\ {}_1''p_*^i, {}_2''c_*^i \end{matrix} \right\} &= \left\{ \begin{matrix} {}_1'\tilde{s}_*^i, \bar{\theta}^3 {}_1'\tilde{s}_*^i \\ {}_1''\tilde{s}_*^i, \bar{\theta}^3 {}_1''\tilde{s}_*^i \end{matrix} \right\}, \quad {}_{AT} \bar{\theta}^3 = \begin{pmatrix} a \\ b \end{pmatrix}; \\ \left\{ \begin{matrix} {}_0'S^i, {}_0\bar{S}^i, {}_0''S^i \\ {}_0'l^i, {}_0\bar{l}^i, {}_0''l^i \end{matrix} \right\} &= \left\{ \begin{matrix} \int_a^b {}_1'\tilde{s}_*^i, \int_c^d {}_1''\tilde{s}_*^i, \int_d^c {}_1''\tilde{s}_*^i \\ \int_b^a {}_1'\tilde{s}_*^i, \int_c^d {}_1''\tilde{s}_*^i, \int_d^c {}_1''\tilde{s}_*^i \end{matrix} \right\} \begin{pmatrix} 1 \\ \bar{\theta}^3 \end{pmatrix} d\bar{\theta}^3. \end{aligned} \quad (3.49)$$

Then, with the above equations and Eqs. (3.28) - (3.34), the boundary conditions are obtained from Eq. (3.48) by applying the fundamental theorem of calculus of variations. The stress boundary conditions are as follows:

on the top surface,

$${}_2'p_*^i - {}_2'p^i = 0, \quad {}_2'c_*^i - {}_2'c^i = 0; \quad (3.50)$$

on the bottom surface,

$${}_1''p_*^i + {}_1''p^i = 0, \quad {}_1''c_*^i + {}_1''c^i = 0; \quad (3.51)$$

on the edge surface,

$$\begin{aligned}
{}_0\dot{S}^\alpha + {}_0\bar{S}^\alpha + {}_0''S^\alpha = {}_0\bar{m}_\beta \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + {}''N^{\beta\gamma})(\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma) \right. \\
+ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}^\alpha|_\gamma + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}^\alpha|_\gamma \\
+ (\bar{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi^\alpha|_\gamma + \dot{Q}^\beta \dot{\psi}^\alpha + \bar{Q}^\beta \bar{\psi}^\alpha + {}''Q^\beta {}''\psi^\alpha \left. \right\}, \quad (3.52)
\end{aligned}$$

$$\begin{aligned}
\frac{\bar{h}}{2} (\dot{S}^\alpha - {}''S^\alpha) + {}_0\bar{\mathcal{L}}^\alpha = {}_0\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] (\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + {}''N^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}^\alpha|_\gamma + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}^\alpha|_\gamma \\
- \frac{\bar{h}}{2} (\bar{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi^\alpha|_\gamma + \frac{\bar{h}}{2} (\dot{Q}^\beta \dot{\psi}^\alpha - {}''Q^\beta {}''\psi^\alpha) + \bar{T}^\beta \bar{\psi}^\alpha \left. \right\}, \quad (3.53)
\end{aligned}$$

$$\begin{aligned}
{}_0\dot{\mathcal{L}}^\alpha - \frac{\bar{h}}{2} \dot{S}^\alpha = {}_0\bar{m}_\beta \left\{ (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma})(\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\psi}^\alpha|_\gamma + (\dot{T}^\beta - \frac{\bar{h}}{2} \dot{Q}^\beta) \dot{\psi}^\alpha \left. \right\}, \quad (3.54)
\end{aligned}$$

$$\begin{aligned}
{}_0''\mathcal{L}^\alpha + \frac{\bar{h}}{2} {}''S^\alpha = {}_0\bar{m}_\beta \left\{ (\bar{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma})(\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 {}''N^{\beta\gamma} + \bar{h} \bar{M}^{\beta\gamma} + \bar{K}^{\beta\gamma} \right] {}''\psi^\alpha|_\gamma + (\bar{T}^\beta + \frac{\bar{h}}{2} {}''Q^\beta) {}''\psi^\alpha \left. \right\}, \quad (3.55)
\end{aligned}$$

$$\begin{aligned}
{}_0\dot{S}^3 + {}_0\bar{S}^3 + {}_0''S^3 = {}_0\bar{m}_\beta \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + {}''N^{\beta\gamma}) \bar{u}_{3,\gamma} \right. \\
+ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}_{3,\gamma} \\
+ (\bar{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi_{3,\gamma} + \dot{Q}^\beta (1 + \dot{\psi}_3) + \bar{Q}^\beta (1 + \bar{\psi}_3) + {}''Q^\beta (1 + {}''\psi_3) \left. \right\}, \quad (3.56)
\end{aligned}$$

$$\begin{aligned}
\frac{\bar{h}}{2} ({}'_S^3 - {}''S^3) + {}'_L^3 = {}_0\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2} ({}'_N^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 ({}'_N^{\beta\gamma} + {}''N^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + \frac{\bar{h}}{2} ({}'_M^{\beta\gamma} - \frac{\bar{h}}{2} {}'_N^{\beta\gamma}) {}'_\psi_{3,\gamma} \\
- \frac{\bar{h}}{2} ({}''M^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) {}''\psi_{3,\gamma} + \frac{\bar{h}}{2} {}'_Q^\beta (1 + {}'_\psi_3) \\
+ \bar{T}^\beta (1 + \bar{\psi}_3) - \frac{\bar{h}}{2} {}''Q^\beta (1 + {}''\psi_3) \left. \right\}, \quad (3.57)
\end{aligned}$$

$$\begin{aligned}
{}_0L^3 - \frac{\bar{h}}{2} {}'_S^3 = {}_0\bar{m}_\beta \left\{ ({}'_M^{\beta\gamma} - \frac{\bar{h}}{2} {}'_N^{\beta\gamma}) (\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 {}'_N^{\beta\gamma} - \bar{h} {}'_M^{\beta\gamma} + {}'_K^{\beta\gamma} \right] {}'_\psi_{3,\gamma} + \left({}'_T^\beta - \frac{\bar{h}}{2} {}'_Q^\beta \right) (1 + {}'_\psi_3) \left. \right\}, \quad (3.58)
\end{aligned}$$

$$\begin{aligned}
{}_0L^3 + \frac{\bar{h}}{2} {}''S^3 = {}_0\bar{m}_\beta \left\{ ({}''M^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) (\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 {}''N^{\beta\gamma} + \bar{h} {}''M^{\beta\gamma} + {}''K^{\beta\gamma} \right] {}''\psi_{3,\gamma} + \left({}''T^\beta + \frac{\bar{h}}{2} {}''Q^\beta \right) (1 + {}''\psi_3) \left. \right\}. \quad (3.59)
\end{aligned}$$

Now, let us establish the displacement boundary conditions which are obtained from Eq. (3.44), i.e.,

$$\int_{{}_0A_v} (v_i - \tilde{v}_i) \delta S_*^i dA = 0.$$

But, from Eq. (3.45), the expanded form is

$$\begin{aligned}
 & \int_{\tau A_V} (\dot{v}_i - \dot{\tilde{v}}_i) \delta \dot{S}_*^i dA + \int_{\theta A_V} (\ddot{v}_i - \ddot{\tilde{v}}_i) \delta \ddot{S}_*^i dA \\
 & + \int_{\varepsilon A_V} (\dot{v}_i - \dot{\tilde{v}}_i) \delta \dot{S}_*^i dA + \int_{\varepsilon \bar{A}_V} (\bar{v}_i - \bar{\tilde{v}}_i) \delta \bar{S}_*^i dA \\
 & + \int_{\varepsilon A_V} (\ddot{v}_i - \ddot{\tilde{v}}_i) \delta \ddot{S}_*^i dA = 0
 \end{aligned} \tag{3.60}$$

for the three layers. By again applying the fundamental theorem of calculus of variations, the displacement boundary conditions become:

on the top surface,

$$\dot{v}_i - \dot{\tilde{v}}_i = 0; \tag{3.61}$$

on the bottom surface,

$$\ddot{v}_i - \ddot{\tilde{v}}_i = 0; \tag{3.62}$$

and on the edge surface,

$$\begin{aligned}
 \dot{v}_i - \dot{\tilde{v}}_i &= 0, \\
 \bar{v}_i - \bar{\tilde{v}}_i &= 0, \\
 \ddot{v}_i - \ddot{\tilde{v}}_i &= 0.
 \end{aligned} \tag{3.63}$$

3.3.3. Strain-displacement equations

The strain-displacement relation, Eq. (2.7), is a direct result of the fourth integral of Eq. (2.2); therefore, consider the equation

$$\int_V [\gamma_{ij} - \frac{1}{2} (v_{i|j} + v_{j|i} + v^r{}_{|i} v_{r|j})] \delta s^{ij} dV = 0. \quad (3.64)$$

The notions employed in Sect. 3.3.1 are also applicable here. First, Eq. (3.64) is written in terms of the three layers of the sandwich plate, i.e.,

$$\begin{aligned} & \int_V [\gamma'_{ij} - \frac{1}{2} (v'_{i|j} + v'_{j|i} + v'^r{}_{|i} v_{r|j})] \delta s'^{ij} dV \\ & + \int_{\bar{V}} [\bar{\gamma}_{ij} - \frac{1}{2} (\bar{v}_{i|j} + \bar{v}_{j|i} + \bar{v}^r{}_{|i} \bar{v}_{r|j})] \delta \bar{s}^{ij} dV \\ & + \int_V [\gamma''_{ij} - \frac{1}{2} (v''_{i|j} + v''_{j|i} + v''^r{}_{|i} v_{r|j})] \delta s''^{ij} dV = 0. \end{aligned} \quad (3.65)$$

Although the integrals can be combined, there is no advantage to do so in this case because $\delta' s^{ij}$, $\delta \bar{s}^{ij}$, $\delta'' s^{ij}$ are independent quantities. Therefore, let us consider just the first integral of Eq. (3.65) and rewrite it by splitting up the tensor as before; then, we obtain

$$\begin{aligned}
& \int_V \left\{ \left[\gamma_{\alpha\beta} - \frac{1}{2} (\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}^{\gamma}_{/\alpha} \dot{v}_{\gamma/\beta} + \dot{v}_{3,\alpha} \dot{v}_{3,\gamma}) \right] \delta s^{\alpha\beta} \right. \\
& + \left[\gamma_{\alpha 3} - \frac{1}{2} (\dot{v}_{\alpha,3} + \dot{v}_{3,\alpha} + \dot{v}^{\gamma}_{/\alpha} \dot{v}_{\gamma,3} + \dot{v}_{3,\alpha} \dot{v}_{3,3}) \right] \delta s^{\alpha 3} \\
& + \left[\gamma_{3\beta} - \frac{1}{2} (\dot{v}_{3,\beta} + \dot{v}_{\beta,3} + \dot{v}^{\gamma}_{,3} \dot{v}_{\gamma/\beta} + \dot{v}_{3,3} \dot{v}_{3,\beta}) \right] \delta s^{3\beta} \\
& \left. + \left[\gamma_{33} - \frac{1}{2} (2 \dot{v}_{3,3} + \dot{v}^{\gamma}_{,3} \dot{v}_{\gamma,3} + \dot{v}_{3,3} \dot{v}_{3,3}) \right] \delta s^{33} \right\} dV.
\end{aligned}
\tag{3.66}$$

The second and third integrals of Eq. (3.65) will be the same except the prime will be replaced by a bar and double prime, respectively.

Now, the fundamental theorem of the calculus of variations can be applied, and the strain-displacement relations for the top layer are

$$\begin{aligned}
\gamma_{\alpha\beta} &= \frac{1}{2} (\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}^{\gamma}_{/\alpha} \dot{v}_{\gamma/\beta} + \dot{v}_{3,\alpha} \dot{v}_{3,\beta}), \\
\gamma_{\alpha 3} &= \frac{1}{2} (\dot{v}_{\alpha,3} + \dot{v}_{3,\alpha} + \dot{v}^{\gamma}_{/\alpha} \dot{v}_{\gamma,3} + \dot{v}_{3,\alpha} \dot{v}_{3,3}), \\
\gamma_{3\beta} &= \frac{1}{2} (\dot{v}_{3,\beta} + \dot{v}_{\beta,3} + \dot{v}^{\gamma}_{,3} \dot{v}_{\gamma/\beta} + \dot{v}_{3,3} \dot{v}_{3,\beta}), \\
\gamma_{33} &= \frac{1}{2} (2 \dot{v}_{3,3} + \dot{v}^{\gamma}_{,3} \dot{v}_{\gamma,3} + \dot{v}_{3,3} \dot{v}_{3,3}).
\end{aligned}
\tag{3.67}$$

Note that

$$\dot{v}^{\gamma}/_{\alpha} \dot{v}_{\gamma,3} = g^{\gamma\delta} \dot{v}_{\delta}/_{\alpha} \dot{v}_{\gamma,3} = \dot{v}_{\delta}/_{\alpha} \dot{v}^{\delta}_{,3} ;$$

therefore, $\dot{\gamma}_{\alpha 3} = \dot{\gamma}_{3\alpha}$ as expected.

By substituting the displacement functions Eqs. (3.14) into Eqs. (3.67), the strain-displacement equations can be written in the form

$$\begin{aligned} \dot{\gamma}_{\alpha\beta} &= \dot{\gamma}_{\alpha\beta} + \bar{\theta}^3 \dot{\gamma}_{\alpha\beta} + (\bar{\theta}^3)^2 \dot{\gamma}_{\alpha\beta} , \\ \dot{\gamma}_{\alpha 3} &= \dot{\gamma}_{3\alpha} = \dot{\gamma}_{\alpha 3} + \bar{\theta}^3 \dot{\gamma}_{\alpha 3} , \\ \dot{\gamma}_{33} &= \dot{\gamma}_{33} \end{aligned} \quad (3.68)$$

where

$$\begin{aligned} \dot{\gamma}_{\alpha\beta} &= \frac{1}{2} \left[\bar{u}_{\alpha}/_{\beta} + \bar{u}_{\beta}/_{\alpha} + \bar{u}^{\gamma}/_{\alpha} \bar{u}_{\gamma}/_{\beta} + \bar{u}_{3,\alpha} \bar{u}_{3,\beta} \right. \\ &\quad + \frac{\bar{h}}{2} (\bar{\psi}_{\alpha}/_{\beta} + \bar{\psi}_{\beta}/_{\alpha} - \dot{\psi}_{\alpha}/_{\beta} - \dot{\psi}_{\beta}/_{\alpha}) \\ &\quad + \frac{\bar{h}}{2} (\bar{\psi}_{\gamma}/_{\beta} - \dot{\psi}_{\gamma}/_{\beta}) \bar{u}^{\gamma}/_{\alpha} + \frac{\bar{h}}{2} (\bar{\psi}^{\gamma}/_{\alpha} - \dot{\psi}^{\gamma}/_{\alpha}) \bar{u}_{\gamma}/_{\beta} \\ &\quad + \frac{\bar{h}}{2} (\bar{\psi}_{3,\beta} - \dot{\psi}_{3,\beta}) \bar{u}_{3,\alpha} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) \bar{u}_{3,\beta} \\ &\quad + \left(\frac{\bar{h}}{2}\right)^2 (\bar{\psi}^{\gamma}/_{\alpha} - \dot{\psi}^{\gamma}/_{\alpha}) (\bar{\psi}_{\gamma}/_{\beta} - \dot{\psi}_{\gamma}/_{\beta}) \\ &\quad \left. + \left(\frac{\bar{h}}{2}\right)^2 (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) (\bar{\psi}_{3,\beta} - \dot{\psi}_{3,\beta}) \right] , \quad (3.69) \end{aligned}$$

$$\begin{aligned}
{}_1\gamma_{\alpha\beta} = & \frac{1}{2} \left[\psi'_{\alpha/\beta} + \psi'_{\beta/\alpha} + \bar{u}'_{\gamma/\alpha} \psi'_{\gamma/\beta} + \bar{u}'_{\gamma/\beta} \psi'_{\gamma/\alpha} \right. \\
& + \bar{u}_{3,\alpha} \psi'_{3,\beta} + \bar{u}_{3,\beta} \psi'_{3,\alpha} + \frac{\hbar}{2} (\bar{\psi}'_{\gamma/\alpha} - \psi'_{\gamma/\alpha}) \psi'_{\gamma/\beta} \\
& + \frac{\hbar}{2} (\bar{\psi}'_{\gamma/\beta} - \psi'_{\gamma/\beta}) \psi'_{\gamma/\alpha} + \frac{\hbar}{2} (\bar{\psi}'_{3,\alpha} - \psi'_{3,\alpha}) \psi'_{3,\beta} \\
& \left. + \frac{\hbar}{2} (\bar{\psi}'_{3,\beta} - \psi'_{3,\beta}) \psi'_{3,\alpha} \right], \quad (3.70)
\end{aligned}$$

$${}_2\gamma_{\alpha\beta} = \frac{1}{2} (\psi'_{\gamma/\alpha} \psi'_{\gamma/\beta} + \psi'_{3,\alpha} \psi'_{3,\beta}), \quad (3.71)$$

$$\begin{aligned}
{}_0\gamma_{\alpha 3} = & \frac{1}{2} \left[\psi'_{\alpha} + \bar{u}_{3,\alpha} + \bar{u}'_{\gamma/\alpha} \psi'_{\gamma} + \bar{u}_{3,\alpha} \psi'_{3} \right. \\
& + \frac{\hbar}{2} (\bar{\psi}'_{3,\alpha} - \psi'_{3,\alpha}) + \frac{\hbar}{2} (\bar{\psi}'_{\gamma/\alpha} - \psi'_{\gamma/\alpha}) \psi'_{\gamma} \\
& \left. + \frac{\hbar}{2} (\bar{\psi}'_{3,\alpha} - \psi'_{3,\alpha}) \psi'_{3} \right], \quad (3.72)
\end{aligned}$$

$${}_1\gamma_{\alpha 3} = \frac{1}{2} (\psi'_{\gamma} \psi'_{\gamma/\alpha} + \psi'_{3} \psi'_{3,\alpha} + \psi'_{3,\alpha}), \quad (3.73)$$

$${}_0\gamma_{33} = \frac{1}{2} (2\psi'_{3} + \psi'_{\gamma} \psi'_{\gamma} + \psi'_{3} \psi'_{3}). \quad (3.74)$$

The strain-displacement equations for the middle and lower layers can be obtained in a similar manner. However, because the equations are rather lengthy, it is sufficient to observe that the equation for ${}''v_{ij}$ can be obtained from the above by substituting a double prime for each prime and a $-\bar{h}$ for each $+\bar{h}$; similarly, the equations for $\bar{v}_{ij}$ are obtained from the above expression for ${}'v_{ij}$ by substituting a dash for each prime and by setting $\bar{h} = 0$.

3.3.4. Stress-strain equations

As before, the stress-strain equation Eq. (2.8) is a direct consequence of the fifth integral of Eq. (2.2); therefore, consider the equation

$$\int_V \left[S^{ij} - \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right) \right] \delta \gamma_{ij} dV = 0. \quad (3.75)$$

In precisely the same manner as before, the integral can be written in terms of the three layers; and, because $\delta'v_{ij}$, $\delta\bar{v}_{ij}$, $\delta''v_{ij}$ are independent quantities, the fundamental theorem of calculus of variation yields the equations

$$\begin{aligned} S^{ij} &= \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma'_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma'_{ji}} \right), \\ \bar{S}^{ij} &= \frac{1}{2} \left(\frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{ij}} + \frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{ji}} \right), \\ {}''S^{ij} &= \frac{1}{2} \left(\frac{\partial {}''\Sigma^*}{\partial {}''\gamma_{ij}} + \frac{\partial {}''\Sigma^*}{\partial {}''\gamma_{ji}} \right), \end{aligned} \quad (3.76)$$

for the upper, middle, and lower layers, respectively. Now, in this work, stress-strain equations in the plastic range are desired. In addition, the results must be consistent with the theory of plasticity for small strains. The following method of obtaining the plasticity equations for finite strain is basically due to Novozhilov [6].

In the modified Hellinger-Reissner theorem, it was assumed that the strain energy density is a function of Green's strain tensor, i.e.,

$$\Sigma^* = \Sigma^*(\gamma_{ij}).$$

If it is further assumed that the material is isotropic as in most theories of plasticity, then the strain energy density can be written in terms of the invariants of the strain tensor

$$\Sigma^* = \Sigma^*(I_1, I_2, I_3)$$

where

$$I_1 = \gamma_k^k,$$

$$I_2 = \frac{1}{2}(\gamma_\ell^k \gamma_k^\ell - \gamma_k^k \gamma_\ell^\ell) = \frac{1}{2}(\gamma_\ell^k \gamma_k^\ell - I_1^2), \quad (3.77)$$

$$\begin{aligned} I_3 &= \frac{1}{6}(2\gamma_\ell^k \gamma_m^\ell \gamma_k^m - 3\gamma_\ell^k \gamma_k^\ell \gamma_m^m + \gamma_k^k \gamma_\ell^\ell \gamma_m^m), \\ &= \frac{1}{3}(\gamma_\ell^k \gamma_m^\ell \gamma_k^m - 3I_1 I_2 - I_1^3). \end{aligned}$$

Then, the first of Eqs. (3.76) can be written as

$$S^{ij} = \frac{\partial \Sigma^*}{\partial I_1} \frac{\partial I_1}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial I_2} \frac{\partial I_2}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial I_3} \frac{\partial I_3}{\partial \gamma_{ij}} \quad (3.78)$$

where the primes have been dropped for convenience. Next, from Eqs. (3.77),

$$\begin{aligned} \frac{\partial I_1}{\partial \gamma_{ij}} &= g^{ij}, \\ \frac{\partial I_2}{\partial \gamma_{ij}} &= \gamma^{ij} - I_1 g^{ij}, \\ \frac{\partial I_3}{\partial \gamma_{ij}} &= \gamma_{\ell}^i \gamma^{\ell j} - I_1 \gamma^{ij} - I_2 g^{ij}. \end{aligned} \quad (3.79)$$

Substituting Eqs. (3.79) into Eq. (3.78) gives

$$\begin{aligned} S^{ij} &= \frac{\partial \Sigma^*}{\partial I_1} g^{ij} + \frac{\partial \Sigma^*}{\partial I_2} (\gamma^{ij} - I_1 g^{ij}) \\ &\quad + \frac{\partial \Sigma^*}{\partial I_3} (\gamma_{\ell}^i \gamma^{\ell j} - I_1 \gamma^{ij} - I_2 g^{ij}). \end{aligned} \quad (3.80)$$

Similar to classical theory, a deviatoric stress tensor can be defined in terms of the Kirchhoff stress tensor as follows:

$$T^{ij} = S^{ij} - \frac{1}{3} S^k_k g^{ij}$$

or

$$T^{ij} = S^{ij} - S g^{ij}$$

where

$$S = \frac{1}{3} S_k^k.$$

Also, in most theories of plasticity a yield criterion is assumed.

We assume a yield criterion of the form

$$T_j^i T_i^j = 2K \quad (3.81)$$

where K is a material parameter depending upon the amount of strain; note that Eq. (3.81) reduces to von Mises' yield criterion for small strain theory.

When the yield criterion Eq. (3.81) is rewritten in terms of the Kirchhoff stress tensor, we obtain

$$S_j^i S_i^j - \frac{1}{3} S_k^k S_l^l = 2K. \quad (3.82)$$

Similar to the strain invariants, there are also stress invariants, the first two of which are

$$J_1 = S_k^k, \quad (3.83)$$

$$J_2 = \frac{1}{2} (S_l^k S_k^l - S_m^m S_p^p) = \frac{1}{2} (S_l^k S_k^l - J_1^2).$$

In terms of the stress invariants, the yield criterion Eq. (3.82) becomes

$$2J_2 + \frac{2}{3} J_1^2 = 2K. \quad (3.84)$$

Also, note that through Eq. (3.80), the yield criterion can be written in terms of the strain energy density and the strain invariants; thus,

$$2 J_2 + \frac{2}{3} J_1^2 = \left(\frac{\partial \Sigma^*}{\partial I_2} \right)^2 \left(2 I_2 + \frac{2}{3} I_1^2 \right) + \frac{\partial \Sigma^*}{\partial I_2} \frac{\partial \Sigma^*}{\partial I_3} \left(6 I_3 + \frac{2}{3} I_2 I_1 \right) + \left(\frac{\partial \Sigma^*}{\partial I_3} \right)^2 \left(\gamma_r^j \gamma_s^r \gamma_t^s \gamma_j^t - 6 I_3 I_1 - \frac{4}{3} I_2^2 - 4 I_2 I_1^2 - I_1^4 \right). \quad (3.85)$$

In addition, substitution of Eq. (3.80) into Eq. (3.83) gives

$$J_1 = 3 \frac{\partial \Sigma^*}{\partial I_1} - 2 I_1 \frac{\partial \Sigma^*}{\partial I_2} - I_2 \frac{\partial \Sigma^*}{\partial I_3}. \quad (3.86)$$

Now, in the deformation theory of plasticity, it is assumed that the strain tensor is proportional to the stress tensor, see Kachanov [20]; therefore, expressions of strain invariants are proportional to corresponding expressions of stress invariants. Eqs. (3.85) and (3.86) can be made consistent with this proportionality requirement if

$$\frac{\partial \Sigma^*}{\partial I_3} = 0, \quad (3.87)$$

and thus become

$$2 J_2 + \frac{2}{3} J_1^2 = \left(\frac{\partial \Sigma^*}{\partial I_2} \right)^2 \left(2 I_2 + \frac{2}{3} I_1^2 \right), \quad (3.88)$$

$$J_1 = 3 \frac{\partial \Sigma^*}{\partial I_1} - 2 I_1 \frac{\partial \Sigma^*}{\partial I_2}. \quad (3.89)$$

Solving Eq. (3.88) for $\partial \Sigma^* / \partial I_2$ yields

$$\frac{\partial \Sigma^*}{\partial I_2} = \frac{\sqrt{2 J_2 + \frac{2}{3} J_1^2}}{\sqrt{2 I_2 + \frac{2}{3} I_1^2}},$$

or

$$\frac{\partial \Sigma^*}{\partial I_2} = 2 \frac{S_0}{\gamma_0} \quad (3.90)$$

where

$$S_0 = \frac{1}{3} (3 S_j^i S_i^j - S_k^k S_l^l)^{\frac{1}{2}},$$

$$\gamma_0 = \frac{2}{3} (3 \gamma_j^i \gamma_i^j - \gamma_k^k \gamma_l^l)^{\frac{1}{2}}. \quad (3.91)$$

The numerical factors in Eqs. (3.91) have been chosen so that in classical theory S_0 and γ_0 reduce to the octahedral shear stress and strain, respectively. Now, Eq. (3.89) can be written as

$$J_1 = 3 \frac{\partial \Sigma^*}{\partial I_1} - 4 I_1 \frac{S_0}{\gamma_0}$$

or

$$\frac{\partial \Sigma^*}{\partial I_1} = \frac{1}{3} (J_1 + 4 I_1 \frac{S_0}{\gamma_0}). \quad (3.92)$$

Thus, the expressions for $\partial \Sigma^* / \partial I_i$ have been determined, and the stress-strain equations can now be established by two different procedures.

For the first procedure, Eqs. (3.87), (3.90) and (3.92) are substituted into Eq. (3.85) to obtain

$$S^{ij} = \frac{1}{3} \left(J_1 + 4 I_1 \frac{S_0}{\gamma_0} \right) g^{ij} + 2 \frac{S_0}{\gamma_0} (\gamma^{ij} - I_1 g^{ij}),$$

or

$$S^{ij} = 2 \frac{S_0}{\gamma_0} \gamma^{ij} + \frac{1}{3} \left(J_1 - 2 \frac{S_0}{\gamma_0} I_1 \right) g^{ij}. \quad (3.93)$$

As already noted, J_1 is proportional to I_1 , i.e.,

$$J_1 = 3k I_1, \quad (3.94)$$

where k is a physical constant of the material and corresponds to the bulk modulus in classical theory. Then, substitution of Eqs. (3.77) and (3.94) into Eq. (3.93) gives

$$S^{ij} = 2 \frac{S_0}{\gamma_0} \gamma^{ij} + \frac{1}{3} \left(3k - 2 \frac{S_0}{\gamma_0} \right) \gamma_\ell^\ell g^{ij}. \quad (3.95)$$

In addition, the inverse relation can be obtained from Eq. (3.95) by using Eqs. (3.77) and (3.94); thus, the expression for the strain tensor in terms of the stress tensor is

$$\gamma^{ij} = \frac{1}{2} \frac{\gamma_0}{S_0} S^{ij} + \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2} \frac{\gamma_0}{S_0} \right) S_\ell^\ell g^{ij}. \quad (3.96)$$

Eq. (3.93) is Hencky's stress-strain equation for finite theory as given by Gleyzal [21].

For the second procedure, it is desirable to have an explicit expression for $\Sigma^*(v_{ij})$ in order to apply the equations resulting from the variational procedure, i.e., Eqs. (3.76). Consider the expressions for $\partial\Sigma^*/\partial I_i$, i.e., Eqs. (3.87), (3.90) and (3.92), which are, respectively,

$$\frac{\partial\Sigma^*}{\partial I_3} = 0,$$

$$\frac{\partial\Sigma^*}{\partial I_2} = 2 \frac{S_0}{\gamma_0},$$

$$\frac{\partial\Sigma^*}{\partial I_1} = \frac{1}{3} \left(J_1 + 4 I_1 \frac{S_0}{\gamma_0} \right).$$

From Eq. (3.94), the last equation can be written as

$$\frac{\partial\Sigma^*}{\partial I_1} = \frac{1}{3} \left(3k + 4 \frac{S_0}{\gamma_0} \right) I_1. \quad (3.97)$$

Then integration of Eqs. (3.87), (3.90) and (3.97) gives, respectively,

$$\Sigma^* = f(I_1, I_2),$$

$$\Sigma^* = 2 \frac{S_0}{\gamma_0} I_2 + f(I_1, I_3),$$

$$\Sigma^* = \frac{1}{6} \left(3k + 4 \frac{S_0}{\gamma_0} \right) I_1^2 + f(I_2, I_3).$$

Together, the above equations imply that

$$\Sigma^* = 2 \frac{S_0}{\gamma_0} I_2 + \frac{1}{6} \left(3k + 4 \frac{S_0}{\gamma_0} \right) I_1^2,$$

or, from Eqs. (3.77),

$$\Sigma^* = \frac{S_0}{\gamma_0} \gamma_\ell^k \gamma_k^\ell + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) \gamma_i^i \gamma_j^j \quad (3.98)$$

which is the desired result. Naturally, Eq. (3.98) can be put in many different forms, such as

$$\Sigma^* = \frac{S_0}{\gamma_0} g^{kr} g^{ls} \gamma_{rl} \gamma_{ks} + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) g^{kr} g^{ls} \gamma_{rk} \gamma_{sl}, \quad (3.99)$$

or

$$\Sigma^* = \frac{S_0}{\gamma_0} (\gamma^{kl} - \frac{1}{3} \gamma_r^r g^{kl}) (\gamma_{kl} - \frac{1}{3} \gamma_s^s g_{kl}) + \frac{k}{2} \gamma_i^i \gamma_j^j. \quad (3.100)$$

Observe that from Eq. (3.99),

$$\begin{aligned} \frac{\partial \Sigma^*}{\partial \gamma_{ij}} &= \frac{S_0}{\gamma_0} g^{kr} g^{ls} (\gamma_{rl} \delta_k^i \delta_s^j + \gamma_{ks} \delta_r^i \delta_\ell^j) \\ &\quad + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) g^{kr} g^{ls} (\gamma_{rk} \delta_\ell^i \delta_s^j + \gamma_{ls} \delta_r^i \delta_k^j), \end{aligned}$$

or

$$\frac{\partial \Sigma^*}{\partial \gamma_{ij}} = \frac{S_0}{\gamma_0} (\gamma^{ij} + \gamma^{ji}) + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) (g^{kr} \gamma_{rk} + g^{ls} \gamma_{sl}) g^{ij}.$$

Because

$$\frac{\partial \Sigma^*}{\partial \gamma_{ij}} = \frac{\partial \Sigma^*}{\partial \gamma_{ji}},$$

$$\gamma_{ij} = \gamma_{ji},$$

the equation

$$S^{ij} = \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right)$$

becomes

$$S^{ij} = 2 \frac{S_0}{\gamma_0} \gamma^{ij} + \frac{1}{3} \left(3k - 2 \frac{S_0}{\gamma_0} \right) \gamma^l_l g^{ij}$$

which is precisely Eq. (3.95).

Later, it will be desirable to have the stress-strain equations in the forms

$$S^{ij} = E^{ijkl} \gamma_{kl},$$

$$\gamma^{ij} = \xi^{ijkl} S_{kl},$$

which will now be developed.

Because of the symmetry of γ^{ij} ,

$$\gamma^{ij} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) \gamma_{kl},$$

and

$$\gamma^k_k = \frac{1}{2} (g^{lr} g^{ks} + g^{ls} g^{kr}) g_{lk} \gamma_{rs}.$$

Therefore, Eq. (3.95) can be written as

$$S^{ij} = \frac{S_0}{\gamma_0} (g^{ik} g^{jl} + g^{il} g^{jk}) \gamma_{kl} \\ + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) (g^{lr} g^{ks} + g^{ks} g^{lr}) g_{lk} g^{ij} \gamma_{rs},$$

or

$$S^{ij} = \frac{S_0}{\gamma_0} (g^{ik} g^{jl} + g^{il} g^{jk}) \gamma_{kl} \\ + \frac{1}{3} (3k - 2 \frac{S_0}{\gamma_0}) g^{ij} g^{rs} \gamma_{rs}.$$

Similarly, because of the symmetry of S^{ij} , Eq. (3.96) can be written as

$$\gamma^{ij} = \frac{1}{4} \frac{S_0}{\gamma_0} (g^{ik} g^{jl} + g^{il} g^{jk}) S_{kl} \\ + \frac{1}{3} (\frac{1}{3k} - \frac{1}{2} \frac{\gamma_0}{S_0}) g^{ij} g^{kl} S_{kl}.$$

Thus, the stress-strain equations become

$$S^{ij} = E^{ijkl} \gamma_{kl}, \quad (3.101)$$

$$\gamma^{ij} = \epsilon^{ijkl} S_{kl}, \quad (3.102)$$

where

$$E^{ijkl} = \frac{S_0}{\gamma_0} (g^{ik} g^{jl} + g^{il} g^{jk}) + \frac{1}{3} (3k - 2 \frac{S_0}{\gamma_0}) g^{ij} g^{kl}, \quad (3.103)$$

$$\xi^{ijkl} = \frac{1}{4} \frac{\gamma_0}{S_0} (g^{ik} g^{jl} + g^{il} g^{jk}) + \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2} \frac{\gamma_0}{S_0} \right) g^{ij} g^{kl}. \quad (3.104)$$

Note the symmetry relations

$$\begin{aligned} E^{ijkl} &= E^{ijlk} = E^{jikl} = E^{kl ij}, \\ \xi^{ijkl} &= \xi^{ijlk} = \xi^{jikl} = \xi^{kl ij}. \end{aligned} \quad (3.105)$$

Because of the preceding work, it is desirable to have the stress-strain equations in terms of subtensors. For example,

$$\begin{aligned} S^{\alpha\beta} &= E^{\alpha\beta\gamma\delta} \gamma_{\gamma\delta} + 2E^{\alpha\beta\gamma 3} \gamma_{\gamma 3} + E^{\alpha\beta 33} \gamma_{33}, \\ S^{\alpha 3} &= E^{\alpha 3\gamma\delta} \gamma_{\gamma\delta} + 2E^{\alpha 3\gamma 3} \gamma_{\gamma 3} + E^{\alpha 333} \gamma_{33}, \\ S^{33} &= E^{33\gamma\delta} \gamma_{\gamma\delta} + 2E^{33\gamma 3} \gamma_{\gamma 3} + E^{3333} \gamma_{33}. \end{aligned}$$

But because of Eqs. (3.103),

$$E^{\alpha\beta\gamma 3} = E^{\alpha 3\gamma\delta} = E^{\alpha 333} = E^{33\gamma 3} = 0. \quad (3.106)$$

Therefore,

$$\begin{aligned} S^{\alpha\beta} &= E^{\alpha\beta\gamma\delta} \gamma_{\gamma\delta} + E^{\alpha\beta 33} \gamma_{33}, \\ S^{\alpha 3} &= 2E^{\alpha 3\gamma 3} \gamma_{\gamma 3}, \end{aligned} \quad (3.107a)$$

$$S^{33} = E^{33\gamma\delta} \gamma_{\gamma\delta} + E^{3333} \gamma_{33} \quad (3.107b)$$

where, from Eq. (3.103),

$$\begin{aligned} E^{\alpha\beta\gamma\delta} &= \frac{S_0}{\gamma_0} (g^{\alpha\delta} g^{\beta\gamma} + g^{\alpha\gamma} g^{\beta\delta}) + \frac{1}{3} (3k - 2 \frac{S_0}{\gamma_0}) g^{\alpha\beta} g^{\gamma\delta}, \\ E^{\alpha\beta 33} &= E^{33\alpha\beta} = \frac{1}{3} (3k - 2 \frac{S_0}{\gamma_0}) g^{\alpha\beta}, \\ E^{\alpha 3 \gamma 3} &= \frac{S_0}{\gamma_0} g^{\alpha\gamma}, \\ E^{3333} &= \frac{4}{3} \frac{S_0}{\gamma_0} + k. \end{aligned} \quad (3.108)$$

From the above, the subtensor form for Eq. (3.102) is obvious.

In this work, it will be assumed that the above stress-strain equations apply to all three layers of the sandwich plate; however, it is not assumed that the material of each layer is the same. Naturally, as indicated by Eqs. (3.76), different stress-strain equations could be applied to each layer which, indeed, would be necessary if, for example, the middle layer is assumed orthotropic as is often the case.

3.4. Buckling Problem

In this work, the bifurcation theory will be used to formulate the buckling problem. The statement of the theorem, quoting from Novozhilov [6], is

The moment of appearance of a possible bifurcation in the solution corresponds to the critical load. Hence, two positions of equilibrium corresponding to an infinitesimal increment in the critical load differ from one another by an infinitesimal amount.

Note that the buckling criteria is associated with two equilibrium positions; therefore, all of the preceding equations are applicable to both positions providing, of course, that the inertia terms σ_{f^i} are omitted. To distinguish between the two positions, a dot will be used. For example, the displacements corresponding to the position which becomes unstable will be denoted by v_i , and the other (stable) position will be denoted by $\dot{v}_i$. Following the procedure of Novozhilov [6], these displacements can be functionally related as follows:

$$\dot{v}_i = v_i + \delta v_i \quad (3.109)$$

where δv_i is the infinitesimal change that occurs. Furthermore, we will assume that the functions v_i are finite and that δ is an infinitesimal quantity which is independent of the coordinates. Naturally, it is assumed that similar relations apply to all quantities; for example, the stresses corresponding to the two equilibrium positions would be related by the equation

$$\dot{S}^{ij} = S^{ij} + \delta s^{ij}. \quad (3.110)$$

Corresponding to Eqs. (3.9), the displacements for the sandwich plate in the stable equilibrium position would be

$$\begin{aligned} \dot{v}_i &= \dot{u}_i + \theta^3 \dot{\psi}_i, \\ \bar{\dot{v}}_i &= \dot{\bar{u}}_i + \bar{\theta}^3 \dot{\bar{\psi}}_i, \\ {}''\dot{v}_i &= {}''\dot{u}_i + {}''\theta^3 {}''\dot{\psi}_i. \end{aligned} \quad (3.111)$$

By applying the interface continuity conditions as before, and by introducing the notations

$$\begin{aligned} \dot{u}_i &= u_i + \delta U_i, \\ \dot{\psi}_i &= \psi_i + \delta \Psi_i, \end{aligned} \quad (3.112)$$

we obtain the following expressions:

$$\begin{aligned} \dot{v}_i &= \bar{u}_i + \frac{\bar{h}}{2} (\bar{\psi}_i - \dot{\psi}_i) + \bar{\theta}^3 \dot{\psi}_i; \\ \bar{\dot{v}}_i &= \bar{u}_i + \bar{\theta}^3 \bar{\psi}_i; \\ {}''\dot{v}_i &= \bar{u}_i - \frac{\bar{h}}{2} (\bar{\psi}_i - {}''\psi_i) + \bar{\theta}^3 {}''\psi_i; \end{aligned} \quad (3.113)$$

which have the same form as Eqs. (3.14).

Now, consider the stress resultants as defined by Eqs. (3.28) - (3.34). For the stable equilibrium position, similar expressions hold; for example,

$$\dot{M}^{\alpha\beta} = \int_b^a \dot{S}^{\alpha\beta} \bar{\theta}^3 d\bar{\theta}^3.$$

By applying Eq. (3.110), an alternate form is

$$\dot{M}^{\alpha\beta} = \int_b^a (\dot{S}^{\alpha\beta} + \delta \dot{S}^{\alpha\beta}) \bar{\theta}^3 d\bar{\theta}^3$$

which from Eq. (3.28) is

$$\dot{M}^{\alpha\beta} = M^{\alpha\beta} + \int_b^a \delta \dot{S}^{\alpha\beta} \bar{\theta}^3 d\bar{\theta}^3.$$

Now, since the relationship between $\dot{M}^{\alpha\beta}$ and $M^{\alpha\beta}$ which is consistent with Eqs. (3.109) and (3.111) is

$$\dot{M}^{\alpha\beta} = M^{\alpha\beta} + \delta m^{\alpha\beta},$$

we find that

$$\delta m^{\alpha\beta} = \int_b^a \delta \dot{S}^{\alpha\beta} \bar{\theta}^3 d\bar{\theta}^3$$

as might be expected. The variation of the other stress resultants are calculated in the same identical manner; the results are as follows:

$$\delta \left\{ \begin{matrix} 'm^{\alpha\beta}, \bar{m}^{\alpha\beta}, ''m^{\alpha\beta} \\ 'm^{\alpha\beta}, \bar{m}^{\alpha\beta}, ''m^{\alpha\beta} \\ 'k^{\alpha\beta}, \bar{k}^{\alpha\beta}, ''k^{\alpha\beta} \end{matrix} \right\} = \delta \left\{ \int_b^a \dot{s}^{\alpha\beta}, \int_c^b \bar{s}^{\alpha\beta}, \int_d^c \ddot{s}^{\alpha\beta} \right\} \left\{ \begin{matrix} 1 \\ \bar{\theta}^3 \\ (\bar{\theta}^3)^2 \end{matrix} \right\} d\bar{\theta}^3; \quad (3.114)$$

$$\delta \left\{ \begin{matrix} 'g^\alpha, \bar{g}^\alpha, ''g^\alpha \\ 't^\alpha, \bar{t}^\alpha, ''t^\alpha \end{matrix} \right\} = \delta \left\{ \int_b^a \dot{s}^{\alpha 3}, \int_c^b \bar{s}^{\alpha 3}, \int_d^c \ddot{s}^{\alpha 3} \right\} \left\{ \begin{matrix} 1 \\ \bar{\theta}^3 \end{matrix} \right\} d\bar{\theta}^3; \quad (3.115)$$

$$\delta \left\{ 'm^{33}, \bar{m}^{33}, ''m^{33} \right\} = \delta \left\{ \int_b^a \dot{s}^{33}, \int_c^b \bar{s}^{33}, \int_d^c \ddot{s}^{33} \right\} d\bar{\theta}^3; \quad (3.116)$$

$$\delta \left\{ \begin{matrix} 'B^i, \bar{B}^i, ''B^i \\ 'W^i, \bar{W}^i, ''W^i \end{matrix} \right\} = \left\{ \begin{matrix} \dot{b}^i - b^i, \bar{\dot{b}}^i - \bar{b}^i, \ddot{b}^i - b^i \\ \dot{w}^i - w^i, \bar{\dot{w}}^i - \bar{w}^i, \ddot{w}^i - w^i \end{matrix} \right\}; \quad (3.117)$$

$$\delta \left\{ \begin{matrix} 'p^i, 'c^i \\ {}^m\bar{p}^i, {}^m\bar{c}^i \\ ''p^i, ''c^i \end{matrix} \right\} = \delta \left\{ \begin{matrix} e^{i3}, \bar{\theta}^3 e^{i3} \\ \bar{e}^{i3}, \bar{\theta}^3 \bar{e}^{i3} \\ e^{i3}, \bar{\theta}^3 e^{i3} \end{matrix} \right\}, \quad AT \left\{ \begin{matrix} \theta^3 \\ \bar{\theta}^3 \\ \theta^3 \end{matrix} \right\} = (-1)^m \left\{ \begin{matrix} h \\ \frac{h}{2} \\ \frac{h}{2} \end{matrix} \right\}; \quad (3.118)$$

$$\delta \left\{ \begin{matrix} 'p^i, 'c^i \\ \bar{p}^i, \bar{c}^i \\ ''p^i, ''c^i \end{matrix} \right\} = \delta \left\{ \begin{matrix} 'B^i + {}_2'p^i - {}_1'p^i, 'W^i + {}_2'c^i - {}_1'c^i \\ \bar{B}^i + {}_2\bar{p}^i - {}_1\bar{p}^i, \bar{W}^i + {}_2\bar{c}^i - {}_1\bar{c}^i \\ ''B^i + {}_2''p^i - {}_1''p^i, ''W^i + {}_2''c^i - {}_1''c^i \end{matrix} \right\}. \quad (3.119)$$

The quantities δe^{i3} are calculated by substituting Eqs. (3.110)

and (3.112) into Eq. (3.25); thus,

$$\begin{aligned}
\dot{E}^{i3} = & (\dot{S}^{33} + \delta \dot{S}^{33}) \left[\delta \dot{\psi}^i + \bar{u}^i /_8 + \delta \bar{U}^i /_8 \right. \\
& + \frac{\bar{h}}{2} (\bar{\psi}^i /_8 + \delta \bar{\Psi}^i /_8 - \dot{\psi}^i /_8 - \delta \dot{\Psi}^i /_8) \\
& + \bar{\theta}^3 (\dot{\psi}^i /_8 + \delta \dot{\Psi}^i /_8) \left. \right] + (\dot{S}^{33} + \delta \dot{S}^{33}) (\delta \dot{\psi}_3^i \\
& + \dot{\psi}^i + \delta \dot{\Psi}^i),
\end{aligned}$$

$$\begin{aligned}
\dot{E}^{i3} = & \dot{E}^{i3} + \delta \dot{S}^{33} \left[\delta \dot{\psi}^i + \bar{u}^i /_8 + \frac{\bar{h}}{2} (\bar{\psi}^i /_8 - \dot{\psi}^i /_8) \right. \\
& + \bar{\theta}^3 \dot{\psi}^i /_8 \left. \right] + \delta \dot{S}^{33} (\delta \dot{\psi}_3^i + \dot{\psi}^i) + \dot{S}^{33} [\delta \bar{U}^i /_8 \\
& + \frac{\bar{h}}{2} (\delta \bar{\Psi}^i /_8 - \delta \dot{\Psi}^i /_8) + \bar{\theta}^3 \delta \dot{\Psi}^i /_8] + \dot{S}^{33} \delta \dot{\Psi}^i \\
& + \text{higher order terms,}
\end{aligned}$$

$$\dot{E}^{i3} = \dot{E}^{i3} + \delta \dot{E}^{i3} + \text{higher order terms.}$$

Therefore,

$$\begin{aligned}
\delta \dot{E}^{i3} = & \delta \dot{S}^{33} \left[\delta \dot{\psi}^i + \bar{u}^i /_8 + \frac{\bar{h}}{2} (\bar{\psi}^i /_8 - \dot{\psi}^i /_8) \right. \\
& + \bar{\theta}^3 \dot{\psi}^i /_8 \left. \right] + \delta \dot{S}^{33} (\delta \dot{\psi}_3^i + \dot{\psi}^i) + \dot{S}^{33} [\delta \bar{U}^i /_8 \\
& + \frac{\bar{h}}{2} (\delta \bar{\Psi}^i /_8 - \delta \dot{\Psi}^i /_8) + \bar{\theta}^3 \delta \dot{\Psi}^i /_8] + \dot{S}^{33} \delta \dot{\Psi}^i,
\end{aligned}$$

(3.120)

and similarly

$$\begin{aligned} \delta \bar{e}^{i3} = & \delta \bar{a}^{\gamma 3} (\delta_{\gamma}^i + \bar{u}^i /_{\gamma} + \bar{\theta}^3 \bar{\psi}^i /_{\gamma}) + \delta \bar{a}^{33} (\delta_3^i + \bar{\psi}^i) \\ & + \bar{S}^{\gamma 3} (\delta \bar{U}^i /_{\gamma} + \bar{\theta}^3 \delta \bar{\Psi}^i /_{\gamma}) + \bar{S}^{33} \delta \bar{\Psi}^i, \quad (3.121) \end{aligned}$$

$$\begin{aligned} \delta \bar{e}^{''i3} = & \delta \bar{a}^{\gamma 3} \left[\delta_{\gamma}^i + \bar{u}^i /_{\gamma} - \frac{\bar{h}}{2} (\bar{\psi}^i /_{\gamma} - \bar{\psi}^i /_{\gamma}) \right. \\ & + \bar{\theta}^3 \bar{\psi}^i /_{\gamma} \left. \right] + \delta \bar{a}^{33} (\delta_3^i + \bar{\psi}^i) + \bar{S}^{\gamma 3} [\delta \bar{U}^i /_{\gamma} \\ & - \frac{\bar{h}}{2} (\delta \bar{\Psi}^i /_{\gamma} - \delta \bar{\Psi}^i /_{\gamma}) + \bar{\theta}^3 \delta \bar{\Psi}^i /_{\gamma}] + \bar{S}^{33} \delta \bar{\Psi}^i. \quad (3.122) \end{aligned}$$

3.4.1. Field equations

With the inertia terms ${}_o f^i$ omitted, Eqs. (3.35) - (3.42) represent the equilibrium equations for the unstable position. By replacing all of the un-dotted quantities ($'N^{\alpha\beta}$, $\bar{u}_{\alpha}$, etc.) by dotted quantities ($\dot{N}^{\alpha\beta}$, $\dot{\bar{u}}_{\alpha}$, etc.), the equations then correspond to the stable equilibrium position. Thus, by using the form of Eq. (3.109), the first equilibrium equation, i.e., Eq. (3.35), for the stable position can be written as

$$\begin{aligned}
& \left\{ (\dot{N}^{\beta\gamma} + \delta \dot{m}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \delta \bar{m}^{\beta\gamma} + \ddot{N}^{\beta\gamma} + \delta \ddot{m}^{\beta\gamma}) (\delta_\gamma^\alpha + \bar{U}^\alpha|_\gamma + \delta \bar{U}^\alpha|_\gamma) \right. \\
& + \left[\frac{\hbar}{2} (\dot{N}^{\beta\gamma} + \delta \dot{m}^{\beta\gamma} - \ddot{N}^{\beta\gamma} - \delta \ddot{m}^{\beta\gamma}) + \bar{M}^{\beta\gamma} + \delta \bar{m}^{\beta\gamma} \right] (\bar{\Psi}^\alpha|_\gamma + \delta \bar{\Psi}^\alpha|_\gamma) \\
& + \left[\dot{M}^{\beta\gamma} + \delta \dot{m}^{\beta\gamma} - \frac{\hbar}{2} (\dot{N}^{\beta\gamma} + \delta \dot{m}^{\beta\gamma}) \right] (\dot{\Psi}^\alpha|_\gamma + \delta \dot{\Psi}^\alpha|_\gamma) \\
& + \left[\ddot{M}^{\beta\gamma} + \delta \ddot{m}^{\beta\gamma} + \frac{\hbar}{2} (\ddot{N}^{\beta\gamma} + \delta \ddot{m}^{\beta\gamma}) \right] (\ddot{\Psi}^\alpha|_\gamma + \delta \ddot{\Psi}^\alpha|_\gamma) \\
& + (\dot{Q}^\beta + \delta \dot{q}^\beta) (\dot{\Psi}^\alpha + \delta \dot{\Psi}^\alpha) + (\bar{Q}^\beta + \delta \bar{q}^\beta) (\bar{\Psi}^\alpha + \delta \bar{\Psi}^\alpha) \\
& + (\ddot{Q}^\beta + \delta \ddot{q}^\beta) (\ddot{\Psi}^\alpha + \delta \ddot{\Psi}^\alpha) \} |_\beta + \dot{p}^\alpha + \delta \dot{p}^\alpha \\
& + \bar{p}^\alpha + \delta \bar{p}^\alpha + \ddot{p}^\alpha + \delta \ddot{p}^\alpha = 0.
\end{aligned} \tag{3.123}$$

Since Eq. (3.35) is still a valid equation that applies to the position that becomes unstable, it can be subtracted from the above equation. Also, higher order terms such as

$$\delta \dot{m}^{\beta\gamma} \delta \bar{U}^\alpha|_\gamma$$

are neglected. Therefore, Eq. (3.123) reduces to

$$\begin{aligned}
& \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + {}''N^{\beta\gamma}) \delta \bar{U}^\alpha /_\gamma + (\delta \dot{m}^{\beta\gamma} + \delta \bar{m}^{\beta\gamma} + \delta {}''m^{\beta\gamma}) (\delta_\gamma^\alpha + \bar{u}^\alpha /_\gamma) \right. \\
& + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - {}''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \delta \bar{\Psi}^\alpha /_\gamma + \left[\frac{\bar{h}}{2} (\delta \dot{m}^{\beta\gamma} - \delta {}''m^{\beta\gamma}) + \delta \bar{m}^{\beta\gamma} \right] \bar{\Psi}^\alpha /_\gamma \\
& + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \delta \bar{\Psi}^\alpha /_\gamma + (\delta \dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \delta \dot{m}^{\beta\gamma}) \dot{\Psi}^\alpha /_\gamma \\
& + (\dot{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}''N^{\beta\gamma}) \delta \bar{\Psi}^\alpha /_\gamma + (\delta {}''m^{\beta\gamma} + \frac{\bar{h}}{2} \delta {}''m^{\beta\gamma}) {}''\Psi^\alpha /_\gamma \\
& + \dot{Q}^\beta \delta \dot{\Psi}^\alpha + \delta \dot{q}^\beta \dot{\Psi}^\alpha + \bar{Q}^\beta \delta \bar{\Psi}^\alpha + \delta \bar{q}^\beta \bar{\Psi}^\alpha \\
& + \dot{Q}^\beta \delta {}''\Psi^\alpha + \delta \dot{q}^\beta {}''\Psi^\alpha \} /_\beta + \delta \dot{P}^\alpha + \delta \bar{P}^\alpha + \delta {}''P^\alpha = 0.
\end{aligned}$$

This same procedure is applied to the remaining equilibrium equations, i.e., Eqs. (3.36) - (3.42); then by cancelling the δ , the buckling equations are obtained as follows:

$$\begin{aligned}
& \{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}^\alpha |_y + (\dot{m}^{\beta\gamma} + \bar{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) (\delta_y^\alpha + \bar{u}^\alpha |_y) \\
& + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}^\alpha |_y + \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{\psi}^\alpha |_y \\
& + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}^\alpha |_y + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\psi}^\alpha |_y + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}^\alpha |_y \\
& + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\psi}^\alpha |_y + \dot{Q}^\beta \dot{\Psi}^\alpha + \bar{Q}^\beta \bar{\Psi}^\alpha + \ddot{Q}^\beta \ddot{\Psi}^\alpha + \dot{q}^\beta \dot{\psi}^\alpha \\
& + \bar{q}^\beta \bar{\psi}^\alpha + \ddot{q}^\beta \ddot{\psi}^\alpha \} |_\beta + \dot{p}^\alpha + \bar{p}^\alpha + \ddot{p}^\alpha = 0; \quad (3.124)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}^\alpha |_y + \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] (\delta_y^\alpha + \bar{u}^\alpha |_y) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}^\alpha |_y + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}^\alpha |_y \\
& + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}^\alpha |_y + \frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\psi}^\alpha |_y \\
& - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}^\alpha |_y - \frac{\bar{h}}{2} (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\psi}^\alpha |_y \\
& + \frac{\bar{h}}{2} (\dot{Q}^\beta \dot{\Psi}^\alpha - \ddot{Q}^\beta \ddot{\Psi}^\alpha + \dot{q}^\beta \dot{\psi}^\alpha - \ddot{q}^\beta \ddot{\psi}^\alpha) \} |_\beta - \bar{Q}^\gamma \bar{U}^\alpha |_y \\
& - \bar{q}^\gamma (\delta_y^\alpha + \bar{u}^\alpha |_y) + (\bar{T}^\beta |_\beta - \bar{N}^{33}) \bar{\Psi}^\alpha + (\bar{t}^\beta |_\beta - \bar{m}^{33}) \bar{\psi}^\alpha \\
& + \frac{\bar{h}}{2} (\dot{p}^\alpha - \ddot{p}^\alpha) + \bar{C}^\alpha = 0; \quad (3.125)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma} \right) (\bar{U}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) (\delta^\alpha_\gamma \right. \\
& + \bar{u}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + K^{\beta\gamma} \right] \dot{\Psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{m}^{\beta\gamma} - \bar{h} \dot{m}^{\beta\gamma} + K^{\beta\gamma} \right] \dot{\psi}^\alpha|_\gamma - \frac{\bar{h}}{2} (\dot{Q}^\beta \dot{\Psi}^\alpha + \dot{g}^\beta \dot{\psi}^\alpha) \Big\} |_\beta \\
& - \dot{Q}^\gamma \left[\bar{U}^\alpha|_\gamma + \frac{\bar{h}}{2} (\bar{\Psi}^\alpha|_\gamma - \dot{\Psi}^\alpha|_\gamma) \right] - \dot{g}^\gamma \left[\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma \right. \\
& + \frac{\bar{h}}{2} (\bar{\psi}^\alpha|_\gamma - \dot{\psi}^\alpha|_\gamma) \Big] + (\dot{T}^\beta|_\beta - \dot{N}^{33}) \dot{\Psi}^\alpha \\
& + (\dot{t}^\beta|_\beta - \dot{m}^{33}) \dot{\psi}^\alpha - \frac{\bar{h}}{2} \dot{\rho}^\alpha + \dot{C}^\alpha = 0 ; \quad (3.126)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma} \right) (\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) (\delta^\alpha_\gamma \right. \\
& + \bar{u}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + K^{\beta\gamma} \right] \ddot{\Psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{m}^{\beta\gamma} + \bar{h} \ddot{m}^{\beta\gamma} + K^{\beta\gamma} \right] \ddot{\psi}^\alpha|_\gamma + \frac{\bar{h}}{2} (\ddot{Q}^\beta \ddot{\Psi}^\alpha + \ddot{g}^\beta \ddot{\psi}^\alpha) \Big\} |_\beta \\
& - \ddot{Q}^\gamma \left[\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} (\bar{\Psi}^\alpha|_\gamma - \ddot{\Psi}^\alpha|_\gamma) \right] - \ddot{g}^\gamma \left[\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma \right. \\
& - \frac{\bar{h}}{2} (\bar{\psi}^\alpha|_\gamma - \ddot{\psi}^\alpha|_\gamma) \Big] + (\ddot{T}^\beta|_\beta - \ddot{N}^{33}) \ddot{\Psi}^\alpha \\
& + (\ddot{t}^\beta|_\beta - \ddot{m}^{33}) \ddot{\psi}^\alpha + \frac{\bar{h}}{2} \ddot{\rho}^\alpha + \ddot{C}^\alpha = 0 ; \quad (3.127)
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}_{3,\gamma} + (\dot{m}^{\beta\gamma} + \bar{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) \bar{u}_{3,\gamma} \right. \\
& + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\Psi}}_{3,\gamma} \\
& + \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\bar{\psi}}_{3,\gamma} \\
& + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\bar{\Psi}}_{3,\gamma} + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\bar{\psi}}_{3,\gamma} \\
& + \dot{Q}^{\beta} \dot{\bar{\Psi}}_3 + \bar{Q}^{\beta} \bar{\Psi}_3 + \ddot{Q}^{\beta} \ddot{\bar{\Psi}}_3 + \dot{\bar{g}}^{\beta} (1 + \dot{\bar{\Psi}}_3) + \bar{g}^{\beta} (1 + \bar{\Psi}_3) \\
& + \ddot{g}^{\beta} (1 + \ddot{\bar{\Psi}}_3) \Big\} |_{\beta} + \dot{\rho}^3 + \bar{\rho}^3 + \ddot{\rho}^3 = 0; \quad (3.128)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}_{3,\gamma} + \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{u}_{3,\gamma} \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\Psi}}_{3,\gamma} \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) + \bar{k}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\bar{\psi}}_{3,\gamma} \\
& - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\bar{\Psi}}_{3,\gamma} - \frac{\bar{h}}{2} (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\bar{\psi}}_{3,\gamma} \\
& + \frac{\bar{h}}{2} \left[\dot{Q}^{\beta} \dot{\bar{\Psi}}_3 - \ddot{Q}^{\beta} \ddot{\bar{\Psi}}_3 + \dot{\bar{g}}^{\beta} (1 + \dot{\bar{\Psi}}_3) - \ddot{g}^{\beta} (1 + \ddot{\bar{\Psi}}_3) \right] |_{\beta} \\
& - \bar{Q}^{\beta} \bar{U}_{3,\beta} - \bar{g}^{\beta} \bar{u}_{3,\beta} + (\bar{T}^{\beta}|_{\beta} - \bar{N}^{33}) \bar{\Psi}_3 + (\bar{t}^{\beta}|_{\beta} - \bar{m}^{33}) (1 + \bar{\Psi}_3) \\
& + \frac{\bar{h}}{2} (\dot{\rho}^3 - \ddot{\rho}^3) + \bar{c}^3 = 0; \quad (3.129)
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma}) + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} \right. \right. \\
& \quad \left. \left. + \dot{K}^{\beta\gamma} \right] \dot{\Psi}_{3,\gamma} + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) (\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\
& \quad \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{m}^{\beta\gamma} - \bar{h} \dot{m}^{\beta\gamma} + \dot{k}^{\beta\gamma} \right] \dot{\psi}_{3,\gamma} - \frac{\bar{h}}{2} [\dot{Q}^\beta \dot{\Psi}_3 \right. \right. \\
& \quad \left. \left. + \dot{g}^\beta (1 + \dot{\psi}_3)] \right\} |_\beta - \dot{Q}^\gamma [\bar{U}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\Psi}_{3,\gamma} - \dot{\Psi}_{3,\gamma})] \\
& \quad - \dot{g}^\gamma [\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma})] + (\dot{T}^\beta|_\beta - \dot{N}^{33}) \dot{\Psi}_3 \\
& \quad + (\dot{t}^\beta|_\beta - \dot{m}^{33}) (1 + \dot{\psi}_3) - \frac{\bar{h}}{2} \dot{\rho}^3 + \dot{C}^3 = 0; \quad (3.130)
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma}) + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} \right. \right. \\
& \quad \left. \left. + \ddot{K}^{\beta\gamma} \right] \ddot{\Psi}_{3,\gamma} + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) (\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\
& \quad \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{m}^{\beta\gamma} + \bar{h} \ddot{m}^{\beta\gamma} + \ddot{k}^{\beta\gamma} \right] \ddot{\psi}_{3,\gamma} + \frac{\bar{h}}{2} [\ddot{Q}^\beta \ddot{\Psi}_3 \right. \right. \\
& \quad \left. \left. + \ddot{g}^\beta (1 + \ddot{\psi}_3)] \right\} |_\beta - \ddot{Q}^\gamma [\bar{U}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\Psi}_{3,\gamma} - \dot{\Psi}_{3,\gamma})] \\
& \quad - \ddot{g}^\gamma [\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma})] + (\ddot{T}^\beta|_\beta - \ddot{N}^{33}) \ddot{\Psi}_3 \\
& \quad + (\ddot{t}^\beta|_\beta - \ddot{m}^{33}) (1 + \ddot{\psi}_3) + \frac{\bar{h}}{2} \ddot{\rho}^3 + \ddot{C}^3 = 0. \quad (3.131)
\end{aligned}$$

3.4.2. Boundary conditions

Here, the general procedure employed is the same as that used in the previous section. All the boundary condition equations of Sect. 3.3.2 are applicable to both equilibrium positions.

First, consider the stress boundary conditions on the top surface which for the unstable position are given by Eqs. (3.51). For the stable position, Eqs. (3.51) become

$${}_2\dot{p}_*^i - {}_2\dot{p}^i = 0 \quad (3.132)$$

$${}_2\dot{c}_*^i - {}_2\dot{c}^i = 0.$$

From Eqs. (3.49) and (3.110),

$${}_2\dot{p}_*^i = \dot{\tilde{S}}_*^i = \tilde{S}_*^i + \delta \tilde{S}_*^i, \quad \text{AT} \quad \bar{\theta}^3 = \frac{\bar{h}}{2} + h,$$

$${}_2\dot{p}^i = \dot{E}^{i3} = E^{i3} + \delta E^{i3}, \quad \text{AT} \quad \bar{\theta}^3 = \frac{\bar{h}}{2} + h,$$

or

$${}_2\dot{p}_*^i = {}_2\dot{p}_*^i + \delta {}_2\dot{p}_*^i, \quad (3.133)$$

$${}_2\dot{p}^i = {}_2\dot{p}^i + \delta {}_2\dot{p}^i,$$

where

$$\delta_2' p_*^i = \delta_2' \tilde{x}_*^i, \quad AT \quad \bar{\theta}^3 = \frac{\bar{h}}{2} + h',$$

$$\delta_2' p^i = \delta_2' e^{i3}, \quad AT \quad \bar{\theta}^3 = \frac{\bar{h}}{2} + h'.$$

By substituting Eqs. (3.133) into Eq. (3.132) and then subtracting Eq. (3.51), we obtain

$$\delta_2' p_*^i - \delta_2' p^i = 0.$$

For convenience, the δ can be cancelled; the stress boundary conditions for the buckling problem become

for the top surface,

$${}_2' p_*^i - {}_2' p^i = 0, \quad {}_2' c_*^i - {}_2' c^i = 0, \quad (3.134)$$

for the bottom surface,

$${}_1'' p_*^i + {}_1'' p^i = 0, \quad {}_1'' c_*^i + {}_1'' c^i = 0, \quad (3.135)$$

and for the edge surface (neglecting the higher order terms),

$$\begin{aligned}
{}_0\dot{\bar{\psi}}^\alpha + {}_0\bar{\psi}^\alpha + {}_0\ddot{\bar{\psi}}^\alpha &= {}_0\bar{m}_\beta \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}^\alpha|_\gamma \right. \\
&+ (\dot{m}^{\beta\gamma} + \bar{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) (\delta_\gamma^\alpha + \bar{u}^\alpha|_\gamma) + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\Psi}}^\alpha|_\gamma \\
&+ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}^\alpha|_\gamma + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\psi}^\alpha|_\gamma \\
&+ \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{\psi}^\alpha|_\gamma + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \ddot{\bar{\Psi}}^\alpha|_\gamma \\
&+ (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\psi}^\alpha|_\gamma + \dot{Q}^\beta \dot{\bar{\Psi}}^\alpha + \bar{Q}^\beta \bar{\Psi}^\alpha + \ddot{Q}^\beta \ddot{\bar{\Psi}}^\alpha \\
&+ \dot{q}^\beta \dot{\psi}^\alpha + \bar{q}^\beta \bar{\psi}^\alpha + \ddot{q}^\beta \ddot{\psi}^\alpha \left. \right\}, \tag{3.136}
\end{aligned}$$

$$\begin{aligned}
\frac{\bar{h}}{2} ({}_0\dot{\bar{\psi}}^\alpha - {}_0\ddot{\bar{\psi}}^\alpha) + {}_0\bar{\bar{\psi}}^\alpha &= {}_0\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}^\alpha|_\gamma \right. \\
&+ \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] (\delta_\gamma^\alpha + \bar{u}^\alpha|_\gamma) + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} \right. \\
&+ \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\bar{\Psi}}^\alpha|_\gamma + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) + \bar{k}^{\beta\gamma} \right] \bar{\bar{\psi}}^\alpha|_\gamma \\
&+ \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\bar{\Psi}}}^\alpha|_\gamma + \frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{m}^{\beta\gamma}) \dot{\bar{\bar{\psi}}}^\alpha|_\gamma \\
&- \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\bar{\bar{\Psi}}}^\alpha|_\gamma - \frac{\bar{h}}{2} (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{m}^{\beta\gamma}) \ddot{\bar{\bar{\psi}}}^\alpha|_\gamma \\
&+ \frac{\bar{h}}{2} (\dot{Q}^\beta \dot{\bar{\bar{\Psi}}}^\alpha - \ddot{Q}^\beta \ddot{\bar{\bar{\Psi}}}^\alpha + \dot{q}^\beta \dot{\bar{\bar{\psi}}}^\alpha - \ddot{q}^\beta \ddot{\bar{\bar{\psi}}}^\alpha) + \bar{T}^\beta \bar{\bar{\Psi}}^\alpha + \bar{\bar{t}}^\beta \bar{\bar{\psi}}^\alpha \left. \right\}; \tag{3.137}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{L}}^\alpha - \frac{\bar{h}}{2} \dot{\mathcal{L}}^\alpha = {}_0\bar{m}_\rho \Big\{ & (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) \\
& + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{n}^{\beta\gamma}) (\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) + (\dot{T}^\beta - \frac{\bar{h}}{2} \dot{Q}^\beta) \dot{\Psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\Psi}^\alpha|_\gamma + (\dot{t}^\beta - \frac{\bar{h}}{2} \dot{g}^\beta) \dot{\psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{m}^{\beta\gamma} - \bar{h} \dot{n}^{\beta\gamma} + \dot{k}^{\beta\gamma} \right] \dot{\psi}^\alpha|_\gamma \Big\}, \tag{3.138}
\end{aligned}$$

$$\begin{aligned}
\ddot{\mathcal{L}}^\alpha + \frac{\bar{h}}{2} \ddot{\mathcal{L}}^\alpha = {}_0\bar{m}_\rho \Big\{ & (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) \\
& + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{n}^{\beta\gamma}) (\delta^\alpha_\gamma + \bar{u}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\psi}^\alpha|_\gamma) + (\ddot{T}^\beta + \frac{\bar{h}}{2} \ddot{Q}^\beta) \ddot{\Psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\Psi}^\alpha|_\gamma + (\ddot{t}^\beta + \frac{\bar{h}}{2} \ddot{g}^\beta) \ddot{\psi}^\alpha|_\gamma \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{m}^{\beta\gamma} + \bar{h} \ddot{n}^{\beta\gamma} + \ddot{k}^{\beta\gamma} \right] \ddot{\psi}^\alpha|_\gamma \Big\}, \tag{3.139}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{L}}^3 + \bar{\mathcal{L}}^3 + \ddot{\mathcal{L}}^3 = {}_0\bar{m}_\rho \Big\{ & (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}_{3,\gamma} + (\dot{n}^{\beta\gamma} + \bar{n}^{\beta\gamma} + \ddot{n}^{\beta\gamma}) \bar{u}_{3,\gamma} \\
& + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + \left[\frac{\bar{h}}{2} (\dot{n}^{\beta\gamma} - \ddot{n}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} \\
& + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}_{3,\gamma} + (\dot{m}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{n}^{\beta\gamma}) \dot{\psi}_{3,\gamma} + \dot{Q}^\beta \dot{\Psi}_3 \\
& + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}_{3,\gamma} + (\ddot{m}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{n}^{\beta\gamma}) \ddot{\psi}_{3,\gamma} + \ddot{Q}^\beta \ddot{\Psi}_3 \\
& + \bar{Q}^\beta \bar{\Psi}_3 + \dot{g}^\beta (1 + \dot{\Psi}_3) + \bar{g}^\beta (1 + \bar{\Psi}_3) + \ddot{g}^\beta (1 + \ddot{\Psi}_3) \Big\}, \tag{3.140}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{L}}^3 - \frac{\hbar}{2} \mathcal{L}^3 = {}_0\bar{m}_\beta \Big\{ & \left[\frac{\hbar}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}_{3,\gamma} \\
& + \left[\frac{\hbar}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \frac{\hbar}{2} (\dot{M}^{\beta\gamma} - \frac{\hbar}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\Psi}}_{3,\gamma} \\
& + \left[\left(\frac{\hbar}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + \frac{\hbar}{2} (\dot{m}^{\beta\gamma} - \frac{\hbar}{2} \dot{m}^{\beta\gamma}) \dot{\psi}_{3,\gamma} \\
& + \left[\left(\frac{\hbar}{2} \right)^2 (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) + \bar{k}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} - \frac{\hbar}{2} (\dot{M}^{\beta\gamma} + \frac{\hbar}{2} \dot{N}^{\beta\gamma}) \dot{\bar{\Psi}}_{3,\gamma} \\
& - \frac{\hbar}{2} (\dot{m}^{\beta\gamma} + \frac{\hbar}{2} \dot{m}^{\beta\gamma}) \dot{\psi}_{3,\gamma} + \bar{T}^\beta \bar{\Psi}_3 + \bar{t}^\beta (1 + \bar{\Psi}_3) \\
& + \frac{\hbar}{2} [\dot{Q}^\beta \dot{\bar{\Psi}}_3 - \ddot{Q}^\beta \ddot{\bar{\Psi}}_3 + \dot{g}^\beta (1 + \dot{\bar{\Psi}}_3) - \ddot{g}^\beta (1 + \ddot{\bar{\Psi}}_3)] \Big\}, \tag{3.141}
\end{aligned}$$

$$\begin{aligned}
\dot{\mathcal{L}}^3 - \frac{\hbar}{2} \mathcal{L}^3 = {}_0\bar{m}_\beta \Big\{ & (\dot{M}^{\beta\gamma} - \frac{\hbar}{2} \dot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} + \frac{\hbar}{2} \bar{\Psi}_{3,\gamma}) \\
& + (\dot{m}^{\beta\gamma} - \frac{\hbar}{2} \dot{m}^{\beta\gamma}) (\bar{u}_{3,\gamma} + \frac{\hbar}{2} \bar{\psi}_{3,\gamma}) + (\dot{T}^\beta - \frac{\hbar}{2} \dot{Q}^\beta) \dot{\bar{\Psi}}_3 \\
& + \left[\left(\frac{\hbar}{2} \right)^2 \dot{N}^{\beta\gamma} - \hbar \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\bar{\Psi}}_{3,\gamma} + (\dot{t}^\beta - \frac{\hbar}{2} \dot{g}^\beta) (1 + \dot{\bar{\Psi}}_3) \\
& + \left[\left(\frac{\hbar}{2} \right)^2 \dot{m}^{\beta\gamma} - \hbar \dot{m}^{\beta\gamma} + \dot{k}^{\beta\gamma} \right] \dot{\psi}_{3,\gamma} \Big\}, \tag{3.142}
\end{aligned}$$

$$\begin{aligned}
\ddot{\mathcal{L}}^3 + \frac{\hbar}{2} \mathcal{L}^3 = {}_0\bar{m}_\beta \Big\{ & (\ddot{M}^{\beta\gamma} + \frac{\hbar}{2} \ddot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} - \frac{\hbar}{2} \bar{\Psi}_{3,\gamma}) \\
& + (\ddot{m}^{\beta\gamma} + \frac{\hbar}{2} \ddot{m}^{\beta\gamma}) (\bar{u}_{3,\gamma} - \frac{\hbar}{2} \bar{\psi}_{3,\gamma}) + (\ddot{T}^\beta + \frac{\hbar}{2} \ddot{Q}^\beta) \ddot{\bar{\Psi}}_3 \\
& + \left[\left(\frac{\hbar}{2} \right)^2 \ddot{N}^{\beta\gamma} + \hbar \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\bar{\Psi}}_{3,\gamma} + (\ddot{t}^\beta + \frac{\hbar}{2} \ddot{g}^\beta) (1 + \ddot{\bar{\Psi}}_3) \\
& + \left[\left(\frac{\hbar}{2} \right)^2 \ddot{m}^{\beta\gamma} + \hbar \ddot{m}^{\beta\gamma} + \ddot{k}^{\beta\gamma} \right] \ddot{\psi}_{3,\gamma} \Big\}, \tag{3.143}
\end{aligned}$$

where

$$\delta \begin{Bmatrix} {}_2'P_*^i, {}_2'\bar{C}_*^i \\ {}_1''P_*^i, {}_1''\bar{C}_*^i \end{Bmatrix} = \delta \begin{Bmatrix} {}_2^{\sim i} s_*, \bar{\theta}^3 {}_2^{\sim i} s_* \\ {}_1^{\sim i} s_*, \bar{\theta}^3 {}_1^{\sim i} s_* \end{Bmatrix}, \quad AT \quad \bar{\theta}^3 = \begin{Bmatrix} \frac{\bar{h}}{2} + h \\ -\frac{\bar{h}}{2} - h \end{Bmatrix}, \quad (3.144)$$

$$\delta \begin{Bmatrix} {}_o' s^i, {}_o^{\sim i} s^i, {}_o'' s^i \\ {}_o' \bar{s}^i, {}_o^{\sim i} \bar{s}^i, {}_o'' \bar{s}^i \end{Bmatrix} = \delta \left\{ \int_b^a {}_2^{\sim i} s_*, \int_c^b {}_1^{\sim i} s_*, \int_d^c {}_1^{\sim i} s_* \right\} \begin{Bmatrix} 1 \\ \bar{\theta}^3 \end{Bmatrix} d\bar{\theta}^3. \quad (3.145)$$

In the equations, the quantities $\delta \tilde{s}_*^i$ are the variations of the prescribed stress vectors that occur during buckling.

Next consider the displacement boundary conditions which for the unstable equilibrium position are given by Eqs. (3.61) - (3.63).

For the stable position, Eq. (3.61) becomes

$$\dot{v}_i - \dot{\tilde{v}}_i = 0,$$

or

$$(\dot{v}_i + \delta \dot{v}_i) - (\dot{\tilde{v}}_i + \delta \dot{\tilde{v}}_i) = 0. \quad (3.146)$$

Subtraction of Eq. (3.61) from Eq. (3.146) yields

$$\delta \dot{v}_i - \delta \dot{\tilde{v}}_i = 0.$$

After cancelling the δ 's, the displacement boundary conditions for the buckling problem become:

on the top surface,

$$\dot{V}_i - \dot{\tilde{V}}_i = 0; \quad (3.147)$$

on the bottom surface,

$$\ddot{V}_i - \ddot{\tilde{V}}_i = 0; \quad (3.148)$$

and on the edge surface,

$$\begin{aligned} \dot{V}_i - \dot{\tilde{V}}_i &= 0, \\ \bar{V}_i - \bar{\tilde{V}}_i &= 0, \end{aligned} \quad (3.149)$$

$$\ddot{V}_i - \ddot{\tilde{V}}_i = 0.$$

In the equations, the quantities $\delta\tilde{V}_i$ are the variations of the prescribed displacements that occur during buckling.

3.43. Variation of strain

Naturally, when the displacements undergo the incremental changes δV_i , the strain tensor also experiences an incremental change. To calculate the incremental change of the strain tensor, first observe that Eqs. (3.67) apply to the unstable position, and the same equations apply to the stable position when all the quantities are dotted. For example, in the stable position, the first of Eqs. (3.67) becomes

$$\dot{\gamma}_{\alpha\beta} = \frac{1}{2} (\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}^{\gamma}/_{\alpha} \dot{v}_{\gamma/\beta} + \dot{v}_{3,\alpha} \dot{v}_{3,\beta}). \quad (3.150)$$

Substituting Eq. (3.109) into Eq. (3.150) gives

$$\begin{aligned} \dot{\gamma}_{\alpha\beta} = \frac{1}{2} [& (\dot{v}_{\alpha} + \delta \dot{V}_{\alpha})/_{\beta} + (\dot{v}_{\beta} + \delta \dot{V}_{\beta})/_{\alpha} \\ & + (\dot{v}^{\gamma} + \delta \dot{V}^{\gamma})/_{\alpha} (\dot{v}_{\gamma} + \delta \dot{V}_{\gamma})/_{\beta} \\ & + (\dot{v}_3 + \delta \dot{V}_3)_{,\alpha} (\dot{v}_3 + \delta \dot{V}_3)_{,\beta}], \end{aligned}$$

or, by using Eq. (3.67),

$$\dot{\gamma}_{\alpha\beta} = \dot{\gamma}_{\alpha\beta} + \delta \dot{\Gamma}_{\alpha\beta} + \text{higher order terms}$$

where

$$\begin{aligned} \delta \dot{\Gamma}_{\alpha\beta} = \frac{1}{2} (& \delta \dot{V}_{\alpha/\beta} + \delta \dot{V}_{\beta/\alpha} + \dot{v}^{\gamma}/_{\alpha} \delta \dot{V}_{\gamma/\beta} + \delta \dot{V}^{\gamma}/_{\alpha} \dot{v}_{\gamma/\beta} \\ & + \dot{v}_{3,\alpha} \delta \dot{V}_{3,\beta} + \delta \dot{V}_{3,\alpha} \dot{v}_{3,\beta}). \end{aligned}$$

Naturally, the δ can be cancelled. Also, through Eqs. (3.14) and (3.113), $\Gamma_{\alpha\beta}$ can be written in terms of the displacement functions $\bar{u}_{\alpha}$, $\bar{\psi}_{\alpha}$, $\bar{\Psi}_{\alpha}$, etc. In this manner, we obtain the following equations:

$$\begin{aligned}
{}'\Gamma_{\alpha\beta} &= {}_0'\Gamma_{\alpha\beta} + \bar{\theta}^3 {}_1'\Gamma_{\alpha\beta} + (\bar{\theta}^3)^2 {}_2'\Gamma_{\alpha\beta} ; \\
{}'\Gamma_{\alpha 3} &= {}'\Gamma_{3\alpha} = {}_0'\Gamma_{\alpha 3} + \bar{\theta}^3 {}_1'\Gamma_{\alpha 3} ; \\
{}'\Gamma_{33} &= {}_0'\Gamma_{33} ;
\end{aligned} \tag{3.151}$$

where

$$\begin{aligned}
{}_0'\Gamma_{\alpha\beta} &= \frac{1}{2} \left\{ \bar{U}_{\alpha/\beta} + \bar{U}_{\beta/\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{\alpha/\beta} + \bar{\Psi}_{\beta/\alpha} - {}'\Psi_{\alpha/\beta} - {}'\Psi_{\beta/\alpha}) \right. \\
&\quad + \left[\bar{u}^{\gamma}_{/\alpha} + \frac{\hbar}{2} (\bar{\psi}^{\gamma}_{/\alpha} - {}'\psi^{\gamma}_{/\alpha}) \right] \left[\bar{U}_{\gamma/\beta} + \frac{\hbar}{2} (\bar{\Psi}_{\gamma/\beta} - {}'\Psi_{\gamma/\beta}) \right] \\
&\quad + \left[\bar{U}^{\gamma}_{/\alpha} + \frac{\hbar}{2} (\bar{\Psi}^{\gamma}_{/\alpha} - {}'\Psi^{\gamma}_{/\alpha}) \right] \left[\bar{u}_{\gamma/\beta} + \frac{\hbar}{2} (\bar{\psi}_{\gamma/\beta} - {}'\psi_{\gamma/\beta}) \right] \\
&\quad + \left[\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\psi}_{3,\alpha} - {}'\psi_{3,\alpha}) \right] \left[U_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - {}'\Psi_{3,\beta}) \right] \\
&\quad \left. + \left[\bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - {}'\Psi_{3,\alpha}) \right] \left[\bar{u}_{3,\beta} + \frac{\hbar}{2} (\bar{\psi}_{3,\beta} - {}'\psi_{3,\beta}) \right] \right\},
\end{aligned} \tag{3.152}$$

$$\begin{aligned}
{}_1\Gamma_{\alpha\beta} = & \frac{1}{2} \left\{ \dot{\Psi}_{\alpha/\beta} + \dot{\Psi}_{\beta/\alpha} + \dot{\Psi}^{\gamma}/_{\alpha} \left[\bar{U}_{\gamma/\beta} + \frac{\hbar}{2} (\bar{\Psi}_{\gamma/\beta} - \dot{\Psi}_{\gamma/\beta}) \right] \right. \\
& + \left[\bar{u}^{\gamma}/_{\alpha} + \frac{\hbar}{2} (\bar{\Psi}^{\gamma}/_{\alpha} - \dot{\Psi}^{\gamma}/_{\alpha}) \right] \dot{\Psi}_{\gamma/\beta} + \dot{\Psi}^{\gamma}/_{\alpha} \left[\bar{u}_{\gamma/\beta} + \frac{\hbar}{2} (\bar{\Psi}_{\gamma/\beta} - \dot{\Psi}_{\gamma/\beta}) \right] \\
& + \left[\bar{U}^{\gamma}/_{\alpha} + \frac{\hbar}{2} (\bar{\Psi}^{\gamma}/_{\alpha} - \dot{\Psi}^{\gamma}/_{\alpha}) \right] \dot{\Psi}_{\gamma/\beta} + \dot{\Psi}_{3,\alpha} \left[\bar{U}_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta}) \right] \\
& + \left[\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right] \dot{\Psi}_{3,\beta} + \dot{\Psi}_{3,\alpha} \left[\bar{u}_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta}) \right] \\
& \left. + \left[\bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right] \dot{\Psi}_{3,\beta} \right\}, \quad (3.153)
\end{aligned}$$

$${}_2\Gamma_{\alpha\beta} = \frac{1}{2} (\dot{\Psi}^{\gamma}/_{\alpha} \dot{\Psi}_{\gamma/\beta} + \dot{\Psi}^{\gamma}/_{\alpha} \dot{\Psi}_{\gamma/\beta} + \dot{\Psi}_{3,\alpha} \dot{\Psi}_{3,\beta} + \dot{\Psi}_{3,\alpha} \dot{\Psi}_{3,\beta}), \quad (3.154)$$

$$\begin{aligned}
{}_0\Gamma_{\alpha 3} = & \frac{1}{2} \left\{ \dot{\Psi}_{\alpha} + \bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right. \\
& + \left[\bar{u}^{\gamma}/_{\alpha} + \frac{\hbar}{2} (\bar{\Psi}^{\gamma}/_{\alpha} - \dot{\Psi}^{\gamma}/_{\alpha}) \right] \dot{\Psi}_{\gamma} + \left[\bar{U}^{\gamma}/_{\alpha} + \frac{\hbar}{2} (\bar{\Psi}^{\gamma}/_{\alpha} - \dot{\Psi}^{\gamma}/_{\alpha}) \right] \dot{\Psi}_{\gamma} \\
& \left. + \left[\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right] \dot{\Psi}_3 + \left[\bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right] \dot{\Psi}_3 \right\}, \quad (3.155)
\end{aligned}$$

$${}_1\Gamma_{\alpha 3} = \frac{1}{2} (\dot{\Psi}_{3,\alpha} + \dot{\Psi}^{\gamma}/_{\alpha} \dot{\Psi}_{\gamma} + \dot{\Psi}^{\gamma}/_{\alpha} \dot{\Psi}_{\gamma} + \dot{\Psi}_{3,\alpha} \dot{\Psi}_3 + \dot{\Psi}_{3,\alpha} \dot{\Psi}_3), \quad (3.156)$$

$${}_0\Gamma_{33} = \dot{\Psi}_3 + \dot{\Psi}^{\gamma} \dot{\Psi}_{\gamma} + \dot{\Psi}_3 \dot{\Psi}_3. \quad (3.157)$$

The above equations can also be applied to the lower layer by replacing the primes by double primes and the $+\bar{h}/2$'s by $-\bar{h}/2$'s. Similarly, for the middle layer, the primes are to be replaced by bars and the $\bar{h}/2$'s are set equal to zero.

3.4.4. Variation of stress

To calculate the variation of the stress tensor δs^{ij} due to the variation of the strain tensor $\delta \Gamma_{ij}$, one can start with Eq. (3.101), i.e.,

$$S^{ij} = E^{ijkl} \gamma_{kl} ;$$

the incremental operation on this equation yields

$$\delta S^{ij} = E^{ijkl} \delta \Gamma_{kl} + \gamma_{kl} \delta E^{ijkl}. \quad (3.158)$$

It follows from Eq. (3.103) that

$$\delta E^{ijkl} = \left(g^{ik} g^{jl} + g^{il} g^{jk} - \frac{2}{3} g^{ij} g^{kl} \right) \delta \left(\frac{S_0}{\gamma_0} \right).$$

Because

$$\begin{aligned} \delta \left(\frac{S_0}{\gamma_0} \right) &= \frac{\gamma_0 \delta S_0 - S_0 \delta \gamma_0}{\gamma_0^2} \\ &= \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) \frac{\delta \gamma_0}{\gamma_0}, \end{aligned}$$

and from Eqs. (3.91)

$$\delta \gamma_o = \frac{4}{g \gamma_o} (3 \gamma_s^r - \gamma_m^m \delta_s^r) \delta \Gamma_r^s,$$

Eq. (3.158) can be written as

$$\begin{aligned} \delta \sigma^{ij} = & E^{ijkl} \delta \Gamma_{kl} + \frac{4}{g \gamma_o^2} \left(\frac{\delta S_o}{\delta \gamma_o} - \frac{S_o}{\gamma_o} \right) (3 \gamma_s^r - \gamma_m^m \delta_s^r) \times \\ & \times (g^{ik} g^{jl} + g^{il} g^{jk} - \frac{2}{3} g^{ij} g^{kl}) \gamma_{kl} \delta \Gamma_r^s. \end{aligned} \quad (3.159)$$

The above is an expression for the stress variation in terms of the strain variation and the strain tensor. However, for most buckling problems, it is more convenient to have the expression in terms of the stress tensor rather than the strain tensor. From Eqs. (3.102) and (3.104), it follows that

$$\gamma_{kl} = \xi_{klmm} S^{mm} \quad (3.160)$$

where

$$\xi_{klmm} = \frac{\gamma_o}{4 S_o} (g_{km} g_{lm} + g_{km} g_{lm}) + \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2} \frac{\gamma_o}{S_o} \right) g_{kl} g_{mm}. \quad (3.161)$$

By substituting Eqs. (3.160) and (3.161) into Eq. (3.159), the equation for the stress variation can be put into the form

$$\delta \sigma^{ij} = M^{ijkl} \delta \Gamma_{kl} \quad (3.162)$$

where

$$M^{ijkl} = E^{ijkl} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{ij} - S g^{ij})(S^{kl} - S g^{kl}). \quad (3.163)$$

Note that M^{ijkl} has the symmetry characteristics

$$M^{ijkl} = M^{ijlk} = M^{jikl} = M^{klij}. \quad (3.164)$$

In classical theory, the quantities

$$\frac{\delta S_0}{\delta \gamma_0}, \quad \frac{S_0}{\gamma_0}$$

are defined as the tangent shear modulus and secant shear modulus, respectively, and can be obtained from a simple stress-strain curve, see Teodosiadis, Langhaar, and Smith [22].

Naturally, the stress variation tensor can be written in terms of subtensors

$$\begin{aligned} \delta \mathcal{L}^{\alpha\beta} &= M^{\alpha\beta\gamma\delta} \delta \Gamma_{\gamma\delta} + 2 M^{\alpha\beta\gamma 3} \delta \Gamma_{\gamma 3} + M^{\alpha\beta 33} \delta \Gamma_{33} \\ \delta \mathcal{L}^{\alpha 3} &= M^{\alpha 3\gamma\delta} \delta \Gamma_{\gamma\delta} + 2 M^{\alpha 3\gamma 3} \delta \Gamma_{\gamma 3} + M^{\alpha 333} \delta \Gamma_{33} \\ \delta \mathcal{L}^{33} &= M^{33\gamma\delta} \delta \Gamma_{\gamma\delta} + 2 M^{33\gamma 3} \delta \Gamma_{\gamma 3} + M^{3333} \delta \Gamma_{33} \end{aligned} \quad (3.165)$$

where, from Eqs. (3.106) and (3.163),

$$M^{\alpha\beta\gamma\delta} = E^{\alpha\beta\gamma\delta} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{\alpha\beta} - Sg^{\alpha\beta}) (S^{\gamma\delta} - Sg^{\gamma\delta}),$$

$$M^{\alpha\beta\gamma 3} = M^{\gamma 3\alpha\beta} = \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{\alpha\beta} - Sg^{\alpha\beta}) S^{\gamma 3},$$

$$M^{\alpha\beta 33} = M^{33\alpha\beta} = E^{\alpha\beta 33} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{\alpha\beta} - Sg^{\alpha\beta}) (S^{33} - S),$$

$$M^{\alpha 3\gamma 3} = E^{\alpha 3\gamma 3} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) S^{\alpha 3} S^{\gamma 3}, \quad (3.166)$$

$$M^{\alpha 333} = M^{33\alpha 3} = \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{33} - S) S^{\alpha 3},$$

$$M^{3333} = E^{3333} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) (S^{33} - S)^2.$$

3.5. Modified Buckling Problem

To solve the buckling problem as it has been formulated, one must first solve the fundamental equations of Sect. 3.3 for the pre-buckling displacement functions $\bar{u}_i, \bar{\psi}_i$, etc.; next, these functions are substituted into the buckling equations of Sect. 3.4, which are then solved. This, obviously, is a complicated and tedious procedure which is rarely performed even for simplified theories, see Bolotin [23]. Instead, the assumption is usually made that the influence of the displacements prior to the loss of stability has a negligible effect on the buckling phenomenon and, therefore, can be neglected. Thus, the buckling problem can be simplified by omitting the displacement functions

$$\bar{u}_i, \bar{\psi}_i, \dot{\psi}_i, \ddot{\psi}_i$$

in the equations of Sect. 3.4.

3.5.1. Field equations

In view of the above, the field equations of Sect. 3.4.1 reduce to the following:

$$\begin{aligned} & \left\{ (\dot{M}^{\alpha\beta} + \bar{M}^{\alpha\beta} + \ddot{M}^{\alpha\beta}) + (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}^\alpha /_\gamma \right. \\ & + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}^\alpha /_\gamma + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}^\alpha /_\gamma \\ & + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}^\alpha /_\gamma + \dot{Q}^\beta \dot{\Psi}^\alpha + \bar{Q}^\beta \bar{\Psi}^\alpha + \ddot{Q}^\beta \ddot{\Psi}^\alpha \left. \right\} /_\beta \\ & + \dot{P}^\alpha + \bar{P}^\alpha + \ddot{P}^\alpha = 0 ; \end{aligned} \quad (3.167)$$

$$\begin{aligned}
& \left\{ \frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta}) + \bar{m}^{\alpha\beta} + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}^\alpha|_\gamma \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}^\alpha|_\gamma + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}^\alpha|_\gamma \\
& - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}^\alpha|_\gamma + \frac{\bar{h}}{2} (\dot{Q}^\beta \dot{\Psi}^\alpha - \ddot{Q}^\beta \ddot{\Psi}^\alpha) \left. \right\} |_\beta \\
& - \bar{Q}^\gamma \bar{U}^\alpha|_\gamma + (\bar{T}^\beta|_\beta - \bar{N}^{33}) \bar{\Psi}^\alpha - \bar{g}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - \ddot{p}^\alpha) + \bar{c}^\alpha = 0;
\end{aligned} \tag{3.168}$$

$$\begin{aligned}
& \left\{ \dot{m}^{\alpha\beta} - \frac{\bar{h}}{2} \dot{m}^{\alpha\beta} + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma + \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \bar{K}^{\beta\gamma} \right] \dot{\Psi}^\alpha|_\gamma - \frac{\bar{h}}{2} \dot{Q}^\beta \dot{\Psi}^\alpha \left. \right\} |_\beta \\
& - \dot{Q}^\gamma \left[\bar{U}^\alpha|_\gamma + \frac{\bar{h}}{2} (\bar{\Psi}^\alpha|_\gamma - \dot{\Psi}^\alpha|_\gamma) \right] + (\dot{T}^\beta|_\beta - \dot{N}^{33}) \dot{\Psi}^\alpha \\
& - \dot{g}^\alpha - \frac{\bar{h}}{2} \dot{p}^\alpha + \dot{c}^\alpha = 0;
\end{aligned} \tag{3.169}$$

$$\begin{aligned}
& \left\{ \ddot{m}^{\alpha\beta} + \frac{\bar{h}}{2} \ddot{m}^{\alpha\beta} + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} \bar{\Psi}^\alpha|_\gamma) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \bar{K}^{\beta\gamma} \right] \ddot{\Psi}^\alpha|_\gamma + \frac{\bar{h}}{2} \ddot{Q}^\beta \ddot{\Psi}^\alpha \left. \right\} |_\beta \\
& - \ddot{Q}^\gamma \left[\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} (\bar{\Psi}^\alpha|_\gamma - \ddot{\Psi}^\alpha|_\gamma) \right] + (\ddot{T}^\beta|_\beta - \ddot{N}^{33}) \ddot{\Psi}^\alpha \\
& - \ddot{g}^\alpha + \frac{\bar{h}}{2} \ddot{p}^\alpha + \ddot{c}^\alpha = 0;
\end{aligned} \tag{3.170}$$

$$\begin{aligned}
& \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}_{3,\gamma} + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} \right. \\
& + \left(\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma} \right) \dot{\Psi}_{3,\gamma} + \left(\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma} \right) \ddot{\Psi}_{3,\gamma} \\
& + \dot{Q}^{\beta} \dot{\Psi}_3 + \bar{Q}^{\beta} \bar{\Psi}_3 + \ddot{Q}^{\beta} \ddot{\Psi}_3 + \dot{g}^{\beta} + \bar{g}^{\beta} + \ddot{g}^{\beta} \left. \right\} |_{\beta} \\
& + \dot{p}^3 + \bar{p}^3 + \ddot{p}^3 = 0; \tag{3.171}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\Psi}_{3,\gamma} \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\Psi}_{3,\gamma} \\
& + \frac{\bar{h}}{2} (\dot{Q}^{\beta} \dot{\Psi}_3 - \ddot{Q}^{\beta} \ddot{\Psi}_3 + \dot{g}^{\beta} - \ddot{g}^{\beta}) \left. \right\} |_{\beta} - \bar{Q}^{\beta} \bar{U}_{3,\beta} \\
& + (\bar{T}^{\beta}|_{\beta} - \bar{N}^{33}) \bar{\Psi}_3 + \bar{t}^{\beta}|_{\beta} - \bar{m}^{33} + \frac{\bar{h}}{2} (\dot{p}^3 - \ddot{p}^3) + \bar{c}^3 = 0; \tag{3.172}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(-\frac{\bar{h}}{2} \dot{N}^{\beta\gamma} + \dot{M}^{\beta\gamma} \right) (\bar{U}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma}) - \frac{\bar{h}}{2} (\dot{Q}^{\beta} \dot{\Psi}_3 + \dot{g}^{\beta}) \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \bar{K}^{\beta\gamma} \right] \dot{\Psi}_{3,\gamma} \left. \right\} |_{\beta} + (\bar{T}^{\beta}|_{\beta} - \dot{N}^{33}) \dot{\Psi}_3 \\
& - \dot{Q}^{\gamma} \left[\bar{U}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\Psi}_{3,\gamma} - \dot{\Psi}_{3,\gamma}) \right] + \bar{t}^{\beta}|_{\beta} - \dot{m}^{33} - \frac{\bar{h}}{2} \dot{p}^3 + \dot{c}^3 = 0; \tag{3.173}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\frac{\bar{h}}{2} \bar{N}^{\beta\gamma} + \bar{M}^{\beta\gamma} \right) \left(\bar{U}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma} \right) + \frac{\bar{h}}{2} \left(\bar{Q}^{\beta} \bar{\Psi}_3 + \bar{J}^{\beta} \right) \right. \\
& \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \bar{N}^{\beta\gamma} + \bar{h} \bar{M}^{\beta\gamma} + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} \right\} |_{\beta} + \left(\bar{T}^{\beta} |_{\beta} - \bar{N}^{33} \right) \bar{\Psi}_3 \\
& - \bar{Q}^{\gamma} \left[\bar{U}_{3,\gamma} - \frac{\bar{h}}{2} \left(\bar{\Psi}_{3,\gamma} - \bar{\Psi}_{3,\gamma} \right) \right] + \bar{t}^{\beta} |_{\beta} - \bar{M}^{33} + \frac{\bar{h}}{2} \bar{P}^3 + \bar{C}^3 = 0.
\end{aligned} \tag{3.174}$$

In these equations, the quantities

$$\bar{N}^{\alpha\beta}, \bar{M}^{\alpha\beta}, \bar{K}^{\alpha\beta}, \text{ etc.}$$

are still defined by Eqs. (3.28) -- (3.34) and (3.114) - (3.119).

However, the quantities δe^{i3} now reduce to the following:

$$\begin{aligned}
\delta e^{i3} &= \delta \bar{e}^{i3} + \bar{S}^{\gamma 3} \left[\delta \bar{U}^i |_{\gamma} + \frac{\bar{h}}{2} \left(\delta \bar{\Psi}^i |_{\gamma} - \delta \bar{\Psi}^i |_{\gamma} \right) \right. \\
& \left. + \bar{\Theta}^3 \delta \bar{\Psi}^i |_{\gamma} \right] + \bar{S}^{33} \delta \bar{\Psi}^i;
\end{aligned} \tag{3.175}$$

$$\delta \bar{e}^{i3} = \delta \bar{e}^{i3} + \bar{S}^{\gamma 3} \left(\delta \bar{U}^i |_{\gamma} + \bar{\Theta}^3 \delta \bar{\Psi}^i |_{\gamma} \right) + \bar{S}^{33} \delta \bar{\Psi}^i; \tag{3.176}$$

$$\begin{aligned}
\delta e^{i3} &= \delta \bar{e}^{i3} + \bar{S}^{\gamma 3} \left[\delta \bar{U}^i |_{\gamma} - \frac{\bar{h}}{2} \left(\delta \bar{\Psi}^i |_{\gamma} - \delta \bar{\Psi}^i |_{\gamma} \right) \right. \\
& \left. + \bar{\Theta}^3 \delta \bar{\Psi}^i |_{\gamma} \right] + \bar{S}^{33} \delta \bar{\Psi}^i.
\end{aligned} \tag{3.177}$$

3.5.2. Boundary conditions

For this case, the equations for the stress boundary conditions on the upper and lower faces are the same as those given in Sec. 3.5.2, i.e.,
on the top surface,

$${}_2P_*^i - {}_2P^i = 0, \quad {}_2C_*^i - {}_2C^i = 0, \quad (3.178)$$

on the bottom surface,

$${}_1P_*^i + {}_1P^i = 0, \quad {}_1C_*^i + {}_1C^i = 0. \quad (3.179)$$

However on the edges, the stress boundary conditions now become

$$\begin{aligned} {}_0\mathcal{L}^\alpha + {}_0\bar{\mathcal{L}}^\alpha + {}_0''\mathcal{L}^\alpha = {}_0\bar{m}_\beta \left\{ {}_0'\mathcal{M}^{\alpha\beta} + {}_0\bar{m}^{\alpha\beta} + {}_0''\mathcal{M}^{\alpha\beta} + \left({}_0'\mathcal{M}^{\beta\gamma} - \frac{\bar{h}}{2} {}_0'\mathcal{N}^{\beta\gamma} \right) {}_0'\Psi^\alpha|_\gamma \right. \\ \left. + \left({}_0'\mathcal{N}^{\beta\gamma} + {}_0\bar{N}^{\beta\gamma} + {}_0''\mathcal{N}^{\beta\gamma} \right) {}_0\bar{U}^\alpha|_\gamma + \left({}_0''\mathcal{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}_0''\mathcal{N}^{\beta\gamma} \right) {}_0''\Psi^\alpha|_\gamma \right. \\ \left. + \left[\frac{\bar{h}}{2} \left({}_0'\mathcal{N}^{\beta\gamma} - {}_0''\mathcal{N}^{\beta\gamma} \right) + {}_0\bar{M}^{\beta\gamma} \right] {}_0\bar{\Psi}^\alpha|_\gamma + {}_0Q^\beta {}_0'\Psi^\alpha + {}_0\bar{Q}^\beta {}_0\bar{\Psi}^\alpha + {}_0Q^\beta {}_0''\Psi^\alpha \right\}, \end{aligned} \quad (3.180)$$

$$\begin{aligned} \frac{\bar{h}}{2} ({}_0'\mathcal{L}^\alpha - {}_0''\mathcal{L}^\alpha) + {}_0\bar{\mathcal{L}}^\alpha = {}_0\bar{m}_\beta \left\{ \frac{\bar{h}}{2} ({}_0'\mathcal{M}^{\alpha\beta} - {}_0''\mathcal{M}^{\alpha\beta}) + {}_0\bar{m}^{\alpha\beta} + \frac{\bar{h}}{2} \left({}_0'\mathcal{M}^{\beta\gamma} - \frac{\bar{h}}{2} {}_0'\mathcal{N}^{\beta\gamma} \right) {}_0'\bar{\Psi}^\alpha|_\gamma \right. \\ \left. + \left[\frac{\bar{h}}{2} \left({}_0'\mathcal{N}^{\beta\gamma} - {}_0''\mathcal{N}^{\beta\gamma} \right) + {}_0\bar{M}^{\beta\gamma} \right] {}_0\bar{U}^\alpha|_\gamma - \frac{\bar{h}}{2} \left({}_0''\mathcal{M}^{\beta\gamma} + \frac{\bar{h}}{2} {}_0''\mathcal{N}^{\beta\gamma} \right) {}_0''\bar{\Psi}^\alpha|_\gamma \right. \\ \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \left({}_0'\mathcal{N}^{\beta\gamma} + {}_0''\mathcal{N}^{\beta\gamma} \right) + {}_0\bar{K}^{\beta\gamma} \right] {}_0\bar{\Psi}^\alpha|_\gamma + \frac{\bar{h}}{2} ({}_0Q^\beta {}_0'\Psi^\alpha - {}_0Q^\beta {}_0''\Psi^\alpha) + {}_0\bar{T}^\beta {}_0\bar{\Psi}^\alpha \right\}, \end{aligned} \quad (3.181)$$

$$\begin{aligned} \dot{\bar{\mathcal{L}}}^\alpha - \frac{\hbar}{2} \dot{\bar{\mathcal{L}}}^\alpha = {}_0\bar{n}_\beta \left\{ \dot{m}^{\alpha\beta} - \frac{\hbar}{2} \dot{m}^{\alpha\beta} + (\dot{M}^{\beta\gamma} - \frac{\hbar}{2} \dot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma + \frac{\hbar}{2} \bar{\Psi}^\alpha|_\gamma) \right. \\ \left. + \left[\left(\frac{\hbar}{2} \right)^2 \dot{N}^{\beta\gamma} - \hbar \dot{M}^{\beta\gamma} + K^{\beta\gamma} \right] \bar{\Psi}^\alpha|_\gamma + (\dot{T}^\beta - \frac{\hbar}{2} \dot{Q}^\beta) \bar{\Psi}^\alpha \right\}, \end{aligned} \quad (3.182)$$

$$\begin{aligned} \ddot{\bar{\mathcal{L}}}^\alpha + \frac{\hbar}{2} \ddot{\bar{\mathcal{L}}}^\alpha = {}_0\bar{n}_\beta \left\{ \ddot{m}^{\alpha\beta} + \frac{\hbar}{2} \ddot{m}^{\alpha\beta} + (\ddot{M}^{\beta\gamma} + \frac{\hbar}{2} \ddot{N}^{\beta\gamma}) (\bar{U}^\alpha|_\gamma - \frac{\hbar}{2} \bar{\Psi}^\alpha|_\gamma) \right. \\ \left. + \left[\left(\frac{\hbar}{2} \right)^2 \ddot{N}^{\beta\gamma} + \hbar \ddot{M}^{\beta\gamma} + K^{\beta\gamma} \right] \bar{\Psi}^\alpha|_\gamma + (\ddot{T}^\beta + \frac{\hbar}{2} \ddot{Q}^\beta) \bar{\Psi}^\alpha \right\}, \end{aligned} \quad (3.183)$$

$$\begin{aligned} \dot{\bar{\mathcal{L}}}^3 + \bar{\mathcal{L}}^3 + \ddot{\bar{\mathcal{L}}}^3 = {}_0\bar{m}_\beta \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}_{3,\gamma} + (\dot{M}^{\beta\gamma} - \frac{\hbar}{2} \dot{N}^{\beta\gamma}) \bar{\Psi}_{3,\gamma} \right. \\ \left. + \left[\frac{\hbar}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + (\ddot{M}^{\beta\gamma} + \frac{\hbar}{2} \ddot{N}^{\beta\gamma}) \bar{\Psi}_{3,\gamma} \right. \\ \left. + \dot{Q}^\beta \bar{\Psi}_3 + \bar{Q}^\beta \bar{\Psi}_3 + \ddot{Q}^\beta \bar{\Psi}_3 + \dot{\mathcal{F}}^\beta + \bar{\mathcal{F}}^\beta + \ddot{\mathcal{F}}^\beta \right\}, \end{aligned} \quad (3.184)$$

$$\begin{aligned} \frac{\hbar}{2} (\dot{\bar{\mathcal{L}}}^3 - \ddot{\bar{\mathcal{L}}}^3) + \bar{\mathcal{L}}^3 = {}_0\bar{n}_\beta \left\{ \left[\frac{\hbar}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}_{3,\gamma} \right. \\ \left. + \left[\left(\frac{\hbar}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} + \frac{\hbar}{2} (\dot{M}^{\beta\gamma} - \frac{\hbar}{2} \dot{N}^{\beta\gamma}) \bar{\Psi}_{3,\gamma} \right. \\ \left. - \frac{\hbar}{2} (\ddot{M}^{\beta\gamma} + \frac{\hbar}{2} \ddot{N}^{\beta\gamma}) \bar{\Psi}_{3,\gamma} + \frac{\hbar}{2} (\dot{Q}^\beta \bar{\Psi}_3 - \ddot{Q}^\beta \bar{\Psi}_3) \right. \\ \left. + \frac{\hbar}{2} (\dot{\mathcal{F}}^\beta - \ddot{\mathcal{F}}^\beta) + \bar{T}^\beta \bar{\Psi}_3 + \bar{\mathcal{L}}^\beta \right\}, \end{aligned} \quad (3.185)$$

$$\begin{aligned}
{}_0\dot{\mathcal{L}}^3 - \frac{\bar{h}}{2} {}_0\dot{\mathcal{L}}^3 = {}_0\bar{m}_\beta \left\{ (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma}) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\Psi}_{3,\gamma} + \dot{t}^\beta - \frac{\bar{h}}{2} \dot{q}^\beta \\
\left. + \left(\dot{T}^\beta - \frac{\bar{h}}{2} \dot{Q}^\beta \right) \dot{\Psi}_3 \right\}, \quad (3.186)
\end{aligned}$$

$$\begin{aligned}
{}_0\ddot{\mathcal{L}}^3 + \frac{\bar{h}}{2} {}_0\ddot{\mathcal{L}}^3 = {}_0\bar{m}_\beta \left\{ (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) (\bar{U}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\gamma}) \right. \\
+ \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\Psi}_{3,\gamma} + \ddot{t}^\beta + \frac{\bar{h}}{2} \ddot{q}^\beta \\
\left. + \left(\ddot{T}^\beta + \frac{\bar{h}}{2} \ddot{Q}^\beta \right) \ddot{\Psi}_3 \right\}. \quad (3.187)
\end{aligned}$$

The definitions of Sects. 3.5.1 and 3.4.2 apply to Eqs. (3.178) - (3.187).

For the displacement boundary conditions, the equation of Sect. 3.4.2, i.e., Eqs. (3.147) - (3.149), hold for this case also.

3.5.3. Variation of strain

By reducing the equations of Sect. 3.4.3, the variations of strain for the upper layer is given by the equations,

$$\begin{aligned}
\delta' \Gamma_{\alpha\beta} &= \delta'_0 \Gamma_{\alpha\beta} + \bar{\theta}^3 \delta'_1 \Gamma_{\alpha\beta} + (\bar{\theta}^3)^2 \delta'_2 \Gamma_{\alpha\beta}, \\
\delta' \Gamma_{\alpha 3} &= \delta' \Gamma_{3\alpha} = \delta'_0 \Gamma_{\alpha 3} + \bar{\theta}^3 \delta'_1 \Gamma_{\alpha 3}, \\
\delta' \Gamma_{33} &= \delta'_0 \Gamma_{33}, \quad (3.188)
\end{aligned}$$

where

$$\begin{aligned}
 {}_0'\Gamma_{\alpha\beta} &= \frac{1}{2} \left[\bar{U}_{\alpha|\beta} + \bar{U}_{\beta|\alpha} + \frac{\bar{h}}{2} (\bar{\Psi}_{\alpha|\beta} + \bar{\Psi}_{\beta|\alpha} - \dot{\Psi}_{\alpha|\beta} - \dot{\Psi}_{\beta|\alpha}) \right], \\
 {}_1'\Gamma_{\alpha\beta} &= \frac{1}{2} (\dot{\Psi}_{\alpha|\beta} + \dot{\Psi}_{\beta|\alpha}), \\
 {}_2'\Gamma_{\alpha\beta} &= 0, \\
 {}_0'\Gamma_{\alpha 3} &= \frac{1}{2} \left[\dot{\Psi}_{\alpha} + \bar{U}_{3,\alpha} + \frac{\bar{h}}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) \right], \\
 {}_1'\Gamma_{\alpha 3} &= \frac{1}{2} \dot{\Psi}_{3,\alpha}, \\
 {}_0'\Gamma_{33} &= \dot{\Psi}_3.
 \end{aligned} \tag{3.189}$$

As before, the equations for the middle layer are obtained from the above by replacing the primes by bars and by letting $\bar{h}/2 = 0$, and the equations for the lower layer are obtained by replacing the primes by double primes and the $+ \bar{h}/2$'s by $- \bar{h}/2$'s.

3.5.4. Variation of stress

The equations of Sect. 3.4.4 remain unchanged and, therefore, apply directly to this case. Naturally, the expressions for the strain tensor variations are to be taken from Sect. 3.5.3.

4. SIMPLIFIED LARGE DEFLECTION THEORY

As was mentioned in Sect. 2, different theories can be developed from the variational principle by assuming a particular form for the strain tensor. In the bending of plates, the displacement component v_3 normal to the plane of the plate is usually much larger than the displacement components v_α in the plane of the plate. For most materials, in fact, the displacements v_α are often infinitesimal quantities; therefore, the nonlinear terms

$$v^\alpha /_i v_\alpha /_j$$

can justifiably be omitted from the strain tensor to obtain

$$\gamma_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i} + v_{3,i} v_{3,j}) \quad (4.1)$$

for the coordinate system developed in Sect. 3.1. Then the variational equation, Eq. (2.1), takes the form

$$\begin{aligned} \int_V \rho_0 ({}_0f^i - {}_0F^i) \delta v_i dV = & \delta \left[\int_V (S^{ij} \gamma_{ij} - \Sigma^*) dV \right. \\ & - \int_V \frac{1}{2} S^{ij} (v_{i,j} + v_{j,i} + v_{3,i} v_{3,j}) dV \\ & \left. + \int_{A_s} \tilde{S}_*^i v_i dA + \int_{A_v} S_*^i (v_i - \tilde{v}_i) dA \right] \end{aligned}$$

which can be put into the form

$$\begin{aligned}
 & \int_V \left\{ [S^{ij}(\delta_i^r + \delta_3^r v_{3,i})] |_{,j} + \rho_0 (F^r - f^r) \right\} \delta v_r dV \\
 & + \int_{A_S} [\tilde{S}_*^i - m_j S^{jr} (\delta_r^i + \delta_3^i v_{3,r})] \delta v_i dA \\
 & + \int_{A_V} (v_i - \tilde{v}_i) \delta S_*^i dA \\
 & + \int_V \left[\gamma_{ij} - \frac{1}{2} (v_i |_{,j} + v_j |_{,i} + v_{3,i} v_{3,j}) \right] \delta S^{ij} dV \\
 & + \int_V \left[S^{ij} - \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right) \right] \delta \gamma_{ij} dV = 0
 \end{aligned} \tag{4.2}$$

from which the fundamental equations are obtained.

4.1. Fundamental Equations

Obviously, the fundamental equations can be obtained from Eq. (4.2) in the same manner that the general equations of Sect. 3.3 were developed from Eq. (2.2). However, through the results already obtained and the variational principle, an alternate method is available, namely, to deduce the equations from those of the general theory. Because the alternate method is instructive and fairly simple for this case, it will be employed.

4.1.1. Equations of motion

From Eq. (4.2), the equations of motion are a result of the equation

$$\int_V \left\{ [S^{ij}(\delta_i^r + \delta_3^r v_{3,i})] |_{,j} + \rho_0 (F^r - f^r) \right\} \delta v_r dV = 0. \quad (4.3)$$

Comparison of Eqs. (3.16) and (4.3) shows that the difference between the equations is the terms

$$\begin{aligned} & \int_V (S^{ij} v^r |_{,i}) |_{,j} \delta v_r dV - \int_V (S^{ij} \delta_3^r v_{3,i}) |_{,j} \delta v_r dV \\ &= \int_V (S^{ij} v^\alpha |_{,i}) |_{,j} \delta v_\alpha dV \end{aligned} \quad (4.4)$$

which must be omitted from the equations of Sect. 3.3.1 to obtain the equations of motion. Note that the terms to be omitted are associated with only the δv_α quantities, not δv_3 ; therefore, not all of the equations of motion will be affected. Formally, the equations are reduced by letting $v^\alpha = 0$ when v^α is multiplied by δv_α . For example, Eqs. (3.18) and (3.19) show that the quantities $A^{\alpha\beta}$ and $B^{\alpha 3}$ become

$$A^{\alpha\beta} = S^{\alpha\beta}, \quad B^{\alpha 3} = S^{\alpha 3},$$

while the quantities $A^{3\beta}$ and B^{33} remain unchanged, i.e.,

$$A^{3\beta} = S^{\beta\gamma} v_{3,\gamma} + S^{\beta 3} (1 + v_{3,3}),$$

$$B^{33} = S^{3\gamma} v_{3,\gamma} + S^{33} (1 + v_{3,3}).$$

When the reduction method is applied to Eq. (3.27), it becomes apparent that the quantities

$$\bar{u}^\alpha, \bar{\psi}^\alpha, \dot{\psi}^\alpha, \ddot{\psi}^\alpha$$

must be set equal to zero in Eqs. (3.35) - (3.38) and Eqs. (3.39) - (3.42) remain unchanged. Therefore, the equations of motions are as follows:

$$(\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta})|_\beta + \dot{p}^\alpha + \bar{p}^\alpha + \ddot{p}^\alpha = \dot{a}^\alpha + \bar{a}^\alpha + \ddot{a}^\alpha; \quad (4.5)$$

$$\begin{aligned} & \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] |_\beta - \bar{Q}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - \ddot{p}^\alpha) + \bar{C}^\alpha \\ & = \frac{\bar{h}}{2} (\dot{a}^\alpha - \ddot{a}^\alpha) + \bar{m}^\alpha; \end{aligned} \quad (4.6)$$

$$(\dot{M}^{\alpha\beta} - \frac{\bar{h}}{2} \dot{N}^{\alpha\beta})|_\beta - \dot{Q}^\alpha - \frac{\bar{h}}{2} \dot{p}^\alpha + \dot{C}^\alpha = -\frac{\bar{h}}{2} \dot{a}^\alpha + \dot{m}^\alpha; \quad (4.7)$$

$$(\ddot{M}^{\alpha\beta} + \frac{\bar{h}}{2} \ddot{N}^{\alpha\beta})|_\beta - \ddot{Q}^\alpha + \frac{\bar{h}}{2} \ddot{p}^\alpha + \ddot{C}^\alpha = \frac{\bar{h}}{2} \ddot{a}^\alpha + \ddot{m}^\alpha; \quad (4.8)$$

$$\begin{aligned}
& \left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{u}_{3,\gamma} + (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}_{3,\gamma} \right. \\
& + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\psi}_{3,\gamma} \\
& + \dot{Q}^\beta (1 + \dot{\psi}_3) + \bar{Q}^\beta (1 + \bar{\psi}_3) + \ddot{Q}^\beta (1 + \ddot{\psi}_3) \Big\} |_\beta \\
& + \dot{p}^3 + \bar{p}^3 + \ddot{p}^3 = \dot{a}^3 + \bar{a}^3 + \ddot{a}^3 ; \tag{4.9}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma}) \dot{\psi}_{3,\gamma} \right. \\
& + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} - \frac{\bar{h}}{2} (\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma}) \ddot{\psi}_{3,\gamma} \\
& + \frac{\bar{h}}{2} \dot{Q}^\beta (1 + \dot{\psi}_3) - \frac{\bar{h}}{2} \ddot{Q}^\beta (1 + \ddot{\psi}_3) \Big\} |_\beta - \bar{Q}^\beta \bar{u}_{3,\beta} \\
& + (\bar{T}^\beta |_\beta - \bar{N}^{33}) (1 + \bar{\psi}_3) + \frac{\bar{h}}{2} (\dot{p}^3 - \ddot{p}^3) + \bar{c}^3 \\
& = \frac{\bar{h}}{2} (\dot{a}^3 - \ddot{a}^3) + \bar{m}^3 ; \tag{4.10}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma} \right) (\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) - \frac{\bar{h}}{2} \dot{Q}^{\beta} (1 + \dot{\psi}_3) \right. \\
& + \left. \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\psi}_{3,\gamma} \right\} |_{\beta} + (\dot{T}^{\beta}|_{\beta} - \dot{N}^{33}) (1 + \dot{\psi}_3) \\
& - \dot{Q}^{\gamma} \left[\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma}) \right] - \frac{\bar{h}}{2} \dot{p}^3 + \dot{c}^3 \\
& = - \frac{\bar{h}}{2} \dot{a}^3 + \dot{m}^3 ; \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left(\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma} \right) (\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) + \frac{\bar{h}}{2} \ddot{Q}^{\beta} (1 + \ddot{\psi}_3) \right. \\
& + \left. \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\psi}_{3,\gamma} \right\} |_{\beta} + (\ddot{T}^{\beta}|_{\beta} - \ddot{N}^{33}) (1 + \ddot{\psi}_3) \\
& - \ddot{Q}^{\gamma} \left[\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \ddot{\psi}_{3,\gamma}) \right] + \frac{\bar{h}}{2} \ddot{p}^3 + \ddot{c}^3 \\
& = \frac{\bar{h}}{2} \ddot{a}^3 + \ddot{m}^3 ; \tag{4.12}
\end{aligned}$$

and the E^{i3} quantities become

$$\begin{aligned}
\dot{E}^{\alpha 3} &= \dot{S}^{\alpha 3}, \quad \bar{E}^{\alpha 3} = \bar{S}^{\alpha 3}, \quad \ddot{E}^{\alpha 3} = \ddot{S}^{\alpha 3}, \\
\dot{E}^{33} &= \dot{S}^{\gamma 3} \left[\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma}) + \bar{\theta}^3 \dot{\psi}_{3,\gamma} \right] + \dot{S}^{33} (1 + \dot{\psi}_3), \\
\bar{E}^{33} &= \bar{S}^{\gamma 3} (\bar{u}_{3,\gamma} + \bar{\theta}^3 \bar{\psi}_{3,\gamma}) + \bar{S}^{33} (1 + \bar{\psi}_3), \\
\ddot{E}^{33} &= \ddot{S}^{\gamma 3} \left[\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \ddot{\psi}_{3,\gamma}) + \bar{\theta}^3 \ddot{\psi}_{3,\gamma} \right] + \ddot{S}^{33} (1 + \ddot{\psi}_3). \tag{4.13}
\end{aligned}$$

4.1.2. Boundary conditions

For the surfaces upon which the stresses are specified, the boundary conditions are determined from the equation

$$\int_{A_s} [\tilde{S}_*^i - {}_0m_j S^{jr} (\delta_r^i + \delta_3^i v_{3,r})] \delta v_i dA = 0 \quad (4.14)$$

which is a result of the second integral of Eq. (4.2). Following the procedure of the preceding section, the difference between Eqs. (3.43) and (4.14) is

$$\begin{aligned} & \int_{A_s} -{}_0m_j S^{jr} v^i|_r \delta v_i dA - \int_{A_s} -{}_0m_j S^{jr} \delta_3^i v_{3,r} \delta v_i dA \\ & = \int_{A_s} -{}_0m_j S^{jr} v^i|_r \delta v_i dA; \end{aligned} \quad (4.15)$$

and, therefore, these terms are to be omitted from the equations of Sect. 3.3.2. Note that the reduction procedure is identically the same as described in the preceding section. Then, with reference to Eq. (3.48), the equations that change are Eqs. (3.52) - (3.55); the stress boundary conditions are as follows:

on the top surface,

$${}_2'p_*^i - {}_2'p^i = 0, \quad {}_2'C_*^i - {}_2'C^i = 0; \quad (4.16)$$

on the bottom surface,

$${}_1''p_*^i + {}_1''p^i = 0, \quad {}_1''C_*^i + {}_1''C^i = 0; \quad (4.17)$$

on the edge surface,

$${}'_0\dot{S}^\alpha + {}'_0\bar{S}^\alpha + {}''_0\dot{S}^\alpha = {}_0\bar{m}_\beta ({}'_0N^{\alpha\beta} + \bar{N}^{\alpha\beta} + {}''_0N^{\alpha\beta}), \quad (4.18)$$

$$\frac{\bar{h}}{2}({}'_0\dot{S}^\alpha - {}''_0\dot{S}^\alpha) + {}'_0\bar{\mathcal{L}}^\alpha = {}_0\bar{m}_\beta \left[\frac{\bar{h}}{2}({}'_0N^{\alpha\beta} - {}''_0N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right], \quad (4.19)$$

$${}'_0\dot{\mathcal{L}}^\alpha - \frac{\bar{h}}{2} {}'_0\dot{S}^\alpha = {}_0\bar{m}_\beta ({}'_0M^{\alpha\beta} - \frac{\bar{h}}{2} {}'_0N^{\alpha\beta}), \quad (4.20)$$

$${}''_0\dot{\mathcal{L}}^\alpha + \frac{\bar{h}}{2} {}''_0\dot{S}^\alpha = {}_0\bar{m}_\beta ({}''_0M^{\alpha\beta} + \frac{\bar{h}}{2} {}''_0N^{\alpha\beta}), \quad (4.21)$$

$$\begin{aligned} {}'_0\dot{S}^3 + {}'_0\bar{S}^3 + {}''_0\dot{S}^3 &= {}_0\bar{m}_\beta \left\{ ({}'_0N^{\beta\gamma} + \bar{N}^{\beta\gamma} + {}''_0N^{\beta\gamma}) \bar{u}_{3,\gamma} \right. \\ &\quad + \left[\frac{\bar{h}}{2}({}'_0N^{\beta\gamma} - {}''_0N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + ({}'_0M^{\beta\gamma} - \frac{\bar{h}}{2} {}'_0N^{\beta\gamma}) {}'_0\psi_{3,\gamma} \\ &\quad \left. + ({}''_0M^{\beta\gamma} + \frac{\bar{h}}{2} {}''_0N^{\beta\gamma}) {}''_0\psi_{3,\gamma} + \bar{Q}^\beta (1 + {}'_0\psi_3) + \bar{Q}^\beta (1 + \bar{\psi}_3) + \bar{Q}^\beta (1 + {}''_0\psi_3) \right\}, \end{aligned} \quad (4.22)$$

$$\begin{aligned} \frac{\bar{h}}{2}({}'_0\dot{S}^3 - {}''_0\dot{S}^3) + {}'_0\bar{\mathcal{L}}^3 &= {}_0\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2}({}'_0N^{\beta\gamma} - {}''_0N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} \right. \\ &\quad + \left[\left(\frac{\bar{h}}{2} \right)^2 ({}'_0N^{\beta\gamma} + {}''_0N^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + \frac{\bar{h}}{2} ({}'_0M^{\beta\gamma} - \frac{\bar{h}}{2} {}'_0N^{\beta\gamma}) {}'_0\psi_{3,\gamma} \\ &\quad - \frac{\bar{h}}{2} ({}''_0M^{\beta\gamma} + \frac{\bar{h}}{2} {}''_0N^{\beta\gamma}) {}''_0\psi_{3,\gamma} + \frac{\bar{h}}{2} \bar{Q}^\beta (1 + {}'_0\psi_3) \\ &\quad \left. + \bar{T}^\beta (1 + \bar{\psi}_3) - \frac{\bar{h}}{2} \bar{Q}^\beta (1 + {}''_0\psi_3) \right\}, \end{aligned} \quad (4.23)$$

$$\begin{aligned} \mathcal{L}^3 - \frac{\bar{h}}{2} \dot{S}^3 = \bar{m}_\beta \left\{ \left(\dot{M}^{\beta\gamma} - \frac{\bar{h}}{2} \dot{N}^{\beta\gamma} \right) (\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\ \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \dot{N}^{\beta\gamma} - \bar{h} \dot{M}^{\beta\gamma} + \dot{K}^{\beta\gamma} \right] \dot{\psi}_{3,\gamma} + \left(\dot{T}^\beta - \frac{\bar{h}}{2} \dot{Q}^\beta \right) (1 + \dot{\psi}_3) \right\}, \end{aligned} \quad (4.24)$$

$$\begin{aligned} \mathcal{L}^3 + \frac{\bar{h}}{2} \ddot{S}^3 = \bar{m}_\beta \left\{ \left(\ddot{M}^{\beta\gamma} + \frac{\bar{h}}{2} \ddot{N}^{\beta\gamma} \right) (\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} \bar{\psi}_{3,\gamma}) \right. \\ \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 \ddot{N}^{\beta\gamma} + \bar{h} \ddot{M}^{\beta\gamma} + \ddot{K}^{\beta\gamma} \right] \ddot{\psi}_{3,\gamma} + \left(\ddot{T}^\beta + \frac{\bar{h}}{2} \ddot{Q}^\beta \right) (1 + \ddot{\psi}_3) \right\}. \end{aligned} \quad (4.25)$$

Also, the equations for E^{i3} , Eqs. (4.13), which were established in the preceding section are applicable to Eqs. (4.16) and (4.17).

The displacement boundary conditions are a result of the third integral of Eq. (4.2); therefore, consider the equation

$$\int_{A_v} (\nu_i - \tilde{\nu}_i) \delta S_*^i dA = 0. \quad (4.26)$$

Comparing the above with Eq. (3.44) shows that the equations are identical, and, thus, the displacement boundary conditions are the same as for the general nonlinear theory, namely,

on the top surface,

$$\dot{\nu}_i - \dot{\tilde{\nu}}_i = 0, \quad (4.27)$$

on the bottom surface,

$$\ddot{\nu}_i - \ddot{\tilde{\nu}}_i = 0, \quad (4.28)$$

on the edge surface,

$$\dot{v}_i - \dot{\tilde{v}}_i = 0, \quad (4.29)$$

$$\bar{v}_i - \bar{\tilde{v}}_i = 0, \quad (4.30)$$

$$''v_i - ''\tilde{v}_i = 0. \quad (4.31)$$

4.1.3. Strain-displacement equations

Obviously, from Eq. (4.2), the terms to be omitted from the general theory are

$$v^\alpha /_i \quad v_\alpha /_j \quad ;$$

that is, all the nonlinear terms with v^α are to be dropped from Eqs. (3.68) - (3.74). Then, the strain-displacement equations for the upper layer become

$$\begin{aligned} \gamma'_{\alpha\beta} &= {}_0\gamma'_{\alpha\beta} + \bar{\theta}^3 {}_1\gamma'_{\alpha\beta} + (\bar{\theta}^3)^2 {}_2\gamma'_{\alpha\beta}, \\ \gamma'_{\alpha 3} &= \gamma'_{3\alpha} = {}_0\gamma'_{\alpha 3} + \bar{\theta}^3 {}_1\gamma'_{\alpha 3}, \\ \gamma'_{33} &= {}_0\gamma'_{33}, \end{aligned} \quad (4.32)$$

where

$$\begin{aligned}
{}_0\gamma_{\alpha\beta} = & \frac{1}{2} \left[\bar{u}_{\alpha/\beta} + \bar{u}_{\beta/\alpha} + \bar{u}_{3,\alpha} \bar{u}_{3,\beta} + \frac{\bar{h}}{2} (\bar{\psi}_{\alpha/\beta} + \bar{\psi}_{\beta/\alpha} \right. \\
& - \dot{\psi}_{\alpha/\beta} - \dot{\psi}_{\beta/\alpha}) + \left(\frac{\bar{h}}{2} \right)^2 (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) (\bar{\psi}_{3,\beta} - \dot{\psi}_{3,\beta}) \\
& \left. + \frac{\bar{h}}{2} \bar{u}_{3,\alpha} (\bar{\psi}_{3,\beta} - \dot{\psi}_{3,\beta}) + \frac{\bar{h}}{2} (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) \bar{u}_{3,\beta} \right], \quad (4.33)
\end{aligned}$$

$$\begin{aligned}
{}_1\gamma_{\alpha\beta} = & \frac{1}{2} \left[\dot{\psi}_{\alpha/\beta} + \dot{\psi}_{\beta/\alpha} + \bar{u}_{3,\alpha} \dot{\psi}_{3,\beta} + \dot{\psi}_{3,\alpha} \bar{u}_{3,\beta} \right. \\
& \left. + \frac{\bar{h}}{2} \dot{\psi}_{3,\alpha} (\bar{\psi}_{3,\beta} - \dot{\psi}_{3,\beta}) + \frac{\bar{h}}{2} (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) \dot{\psi}_{3,\beta} \right], \quad (4.34)
\end{aligned}$$

$${}_2\gamma_{\alpha\beta} = \frac{1}{2} \dot{\psi}_{3,\alpha} \dot{\psi}_{3,\beta}, \quad (4.35)$$

$$\begin{aligned}
{}_0\gamma_{\alpha 3} = & \frac{1}{2} \left[\dot{\psi}_{\alpha} + \bar{u}_{3,\alpha} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) \right. \\
& \left. + \bar{u}_{3,\alpha} \dot{\psi}_3 + \frac{\bar{h}}{2} (\bar{\psi}_{3,\alpha} - \dot{\psi}_{3,\alpha}) \dot{\psi}_3 \right], \quad (4.36)
\end{aligned}$$

$${}_1\gamma_{\alpha 3} = \frac{1}{2} (\dot{\psi}_{3,\alpha} + \dot{\psi}_{3,\alpha} \dot{\psi}_3), \quad (4.37)$$

$${}_0\gamma_{33} = \frac{1}{2} (2 \dot{\psi}_3 + \dot{\psi}_3 \dot{\psi}_3). \quad (4.38)$$

The strain-displacement equations for the middle and lower layers are obtained from the above equations by the procedure described in Sect. 3.3.3.

4.1.4. Stress-strain equations

The derivation of the stress-strain equations that was developed in Sect. 3.3.4 is also applicable to this case; however, in using those equations, the expressions for the strains are to be taken, naturally, from Sect. 4.1.3, not Sect. 3.3.3. For example, the stress $'S^{\alpha\beta}$ is given by Eq. (3.107) as

$$'S^{\alpha\beta} = 'E^{\alpha\beta\gamma\delta} '\gamma_{\gamma\delta} + 'E^{\alpha\beta 33} '\gamma_{33}$$

where

$$'E^{\alpha\beta\gamma\delta} = \frac{'\dot{S}_0}{'\gamma_0} (g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}) + \frac{1}{3} (3k' - 2 \frac{'\dot{S}_0}{'\gamma_0}) g^{\alpha\beta} g^{\gamma\delta},$$

$$'E^{\alpha\beta 33} = \frac{1}{3} (3k' - 2 \frac{'\dot{S}_0}{'\gamma_0}) g^{\alpha\beta},$$

$$'S_0 = \frac{1}{3} (3 'S_j^i 'S_i^j - 'S_k^k 'S_l^l)^{\frac{1}{2}},$$

$$' \gamma_0 = \frac{2}{3} (3 '\gamma_j^i '\gamma_i^j - '\gamma_k^k '\gamma_l^l)^{\frac{1}{2}},$$

and $' \gamma_{\gamma\delta}, ' \gamma_{33}$ are given by Eqs. (4.32).

4.2. Buckling Problem

Naturally, the equations associated with the buckling problem can be developed in the same manner that the equations for the general nonlinear theory were developed in Sect. 3.4. However, in order to be consistent with the preceding section, the buckling equations will be deduced from the general nonlinear theory.

4.2.1. Field equations

The field equations are a direct consequence of the equilibrium equations (i.e., the equations of motion with the inertia terms omitted); therefore, the discussion of Sect. 4.1.1 is also applicable to this case. Thus, the first four field equations are obtained from the general equations by omitting all of the v^α terms. Now, from Eq. (3.109), the omission of the v^α terms also implies the omission of the v^σ terms; therefore, by direct inspection of Eqs. (3.124) - (3.127), the first four field equations become

$$(\dot{m}^{\alpha\beta} + \bar{m}^{\alpha\beta} + {}''m^{\alpha\beta})|_\beta + \dot{p}^\alpha + \bar{p}^\alpha + {}''p^\alpha = 0, \quad (4.39)$$

$$\left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - {}''m^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right] |_\beta - \bar{g}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - {}''p^\alpha) + \bar{c}^\alpha = 0, \quad (4.40)$$

$$(\dot{m}^{\alpha\beta} - \frac{\bar{h}}{2} \dot{m}^{\alpha\beta})|_\beta - \dot{g}^\alpha - \frac{\bar{h}}{2} \dot{p}^\alpha + \dot{c}^\alpha = 0, \quad (4.41)$$

$$({}''m^{\alpha\beta} + \frac{\bar{h}}{2} {}''m^{\alpha\beta})|_\beta - {}''g^\alpha + \frac{\bar{h}}{2} {}''p^\alpha + {}''c^\alpha = 0. \quad (4.42)$$

The other four field equations are the same as for general nonlinear theory and are given by Eqs. (3.128) - (3.131).

In the field equations, the stress resultants are defined by Eqs. (3.28) - (3.34) and the variation of stress resultants by Eqs. (3.114) - (3.119). In addition, the quantities δe^{i3} now reduce to

$$\begin{aligned}\delta e^{\alpha 3} &= \delta \bar{u}^{\alpha 3}, \quad \delta \bar{e}^{\alpha 3} = \delta \bar{u}^{\alpha 3}, \quad \delta e^{\alpha 3} = \delta \bar{u}^{\alpha 3}, \\ \delta e^{33} &= \delta \bar{u}^{\gamma 3} \left[\bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \dot{\psi}_{3,\gamma}) + \bar{\theta}^3 \dot{\psi}_{3,\gamma} \right] \\ &\quad + \delta \bar{u}^{33} (1 + \dot{\psi}_3) + \dot{S}^{\gamma 3} \left[\delta \bar{u}_{3,\gamma} + \frac{\bar{h}}{2} (\delta \bar{\psi}_{3,\gamma} - \delta \dot{\psi}_{3,\gamma}) \right. \\ &\quad \left. + \bar{\theta}^3 \delta \dot{\psi}_{3,\gamma} \right] + \dot{S}^{33} \delta \dot{\psi}_3, \quad (4.43)\end{aligned}$$

$$\begin{aligned}\delta \bar{e}^{33} &= \delta \bar{u}^{\gamma 3} (\bar{u}_{3,\gamma} + \bar{\theta}^3 \bar{\psi}_{3,\gamma}) + \delta \bar{u}^{33} (1 + \bar{\psi}_3) \\ &\quad + \bar{S}^{\gamma 3} (\delta \bar{u}_{3,\gamma} + \bar{\theta}^3 \delta \bar{\psi}_{3,\gamma}) + \bar{S}^{33} \delta \bar{\psi}_3, \\ \delta e^{\alpha 3} &= \delta \bar{u}^{\gamma 3} \left[\bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\bar{\psi}_{3,\gamma} - \ddot{\psi}_{3,\gamma}) + \bar{\theta}^3 \ddot{\psi}_{3,\gamma} \right] \\ &\quad + \delta \bar{u}^{33} (1 + \ddot{\psi}_3) + \ddot{S}^{\gamma 3} \left[\delta \bar{u}_{3,\gamma} - \frac{\bar{h}}{2} (\delta \bar{\psi}_{3,\gamma} - \delta \ddot{\psi}_{3,\gamma}) \right. \\ &\quad \left. + \bar{\theta}^3 \delta \ddot{\psi}_{3,\gamma} \right] + \ddot{S}^{33} \delta \ddot{\psi}_3.\end{aligned}$$

Hence, Eqs. (4.47) instead of Eqs. (3.120) - (3.122) are to be used in the definitions of P^i , Eqs. (3.119).

4.2.2. Boundary conditions

As with the field equations, only the form of Eqs. (3.136)-(3.139) change; the other equations remain unchanged.

Therefore, the stress boundary conditions for the top and bottom surfaces are given by Eqs. (3.134) and (3.135); while for the edge surfaces, the equations are

$${}_0\dot{\varepsilon}^\alpha + {}_0\bar{\varepsilon}^\alpha + {}_0''\varepsilon^\alpha = {}_0\bar{m}_\beta \left({}_0'\dot{m}^{\alpha\beta} + {}_0\bar{m}^{\alpha\beta} + {}_0''\dot{m}^{\alpha\beta} \right), \quad (4.44)$$

$$\frac{\bar{h}}{2} ({}_0\dot{\varepsilon}^\alpha - {}_0''\varepsilon^\alpha) + {}_0\bar{\varepsilon}^\alpha = {}_0\bar{m}_\beta \left[\frac{\bar{h}}{2} ({}_0'\dot{m}^{\alpha\beta} - {}_0''\dot{m}^{\alpha\beta}) + {}_0\bar{m}^{\alpha\beta} \right], \quad (4.45)$$

$${}_0\dot{\varepsilon}^\alpha - \frac{\bar{h}}{2} {}_0\dot{\varepsilon}^\alpha = {}_0\bar{m}_\beta \left({}_0'\dot{m}^{\alpha\beta} - \frac{\bar{h}}{2} {}_0'\dot{m}^{\alpha\beta} \right), \quad (4.46)$$

$${}_0''\varepsilon^\alpha + \frac{\bar{h}}{2} {}_0''\varepsilon^\alpha = {}_0\bar{m}_\beta \left({}_0''\dot{m}^{\alpha\beta} + \frac{\bar{h}}{2} {}_0''\dot{m}^{\alpha\beta} \right), \quad (4.47)$$

and Eqs. (3.140) - (3.143). As before, Eqs. (4.43) which define the quantities δe^{i3} are to be used in Eqs. (3.119).

The displacement boundary conditions remain unchanged and are given by Eqs. (3.147) - (3.149).

4.2.3. Variation of strain

Obviously, the change here is to omit all of the nonlinear terms involving either v^α or V^α from Eq. (3.150). Hence, the variation of strain for the upper layer is given by the equations as follows:

$$\delta' \Gamma_{\alpha\beta} = \delta_0' \Gamma_{\alpha\beta} + \bar{\theta}^3 \delta_1' \Gamma_{\alpha\beta} + (\bar{\theta}^3)^2 \delta_2' \Gamma_{\alpha\beta} ;$$

$$\delta' \Gamma_{\alpha 3} = \delta_0' \Gamma_{\alpha 3} + \bar{\theta}^3 \delta_1' \Gamma_{\alpha 3} ; \quad (4.48)$$

$$\delta' \Gamma_{33} = \delta_0' \Gamma_{33} ;$$

where

$$\begin{aligned} \delta_0' \Gamma_{\alpha\beta} = & \frac{1}{2} \left\{ \bar{U}_\alpha /_\beta + \bar{U}_\beta /_\alpha + \frac{\hbar}{2} (\bar{\Psi}_\alpha /_\beta + \bar{\Psi}_\beta /_\alpha - \dot{\Psi}_\alpha /_\beta - \dot{\Psi}_\beta /_\alpha) \right. \\ & + [\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] [\bar{U}_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta})] \\ & \left. + [\bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] [\bar{u}_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta})] \right\}, \end{aligned} \quad (4.49)$$

$$\begin{aligned} \delta_1' \Gamma_{\alpha\beta} = & \frac{1}{2} \left\{ \dot{\Psi}_\alpha /_\beta + \dot{\Psi}_\beta /_\alpha + \dot{\Psi}_{3,\alpha} [\bar{U}_{3,\beta} + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta})] \right. \\ & + [\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] \dot{\Psi}_{3,\beta} + \dot{\Psi}_{3,\alpha} [\bar{u}_{3,\beta} \\ & \left. + \frac{\hbar}{2} (\bar{\Psi}_{3,\beta} - \dot{\Psi}_{3,\beta})] + [\bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] \dot{\Psi}_{3,\beta} \right\}, \end{aligned} \quad (4.50)$$

$$\delta_2' \Gamma_{\alpha\beta} = \frac{1}{2} (\dot{\Psi}_{3,\alpha} \dot{\Psi}_{3,\beta} + \dot{\Psi}_{3,\alpha} \dot{\Psi}_{3,\beta}), \quad (4.51)$$

$$\begin{aligned} \delta_0' \Gamma_{\alpha 3} = & \frac{1}{2} \left\{ \dot{\Psi}_\alpha + \bar{U}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha}) + \dot{\Psi}_3 [\bar{U}_{3,\alpha} \right. \\ & \left. + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] + \dot{\Psi}_3 [\bar{u}_{3,\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{3,\alpha} - \dot{\Psi}_{3,\alpha})] \right\}, \end{aligned} \quad (4.52)$$

$${}_1\bar{\Gamma}_{\alpha 3} = \frac{1}{2} \left(\bar{\Psi}_{3,\alpha} + \psi_3 \bar{\Psi}_{3,\alpha} + \bar{\Psi}_3 \psi_{3,\alpha} \right), \quad (4.53)$$

$${}_0\bar{\Gamma}_{33} = \bar{\Psi}_3 + \psi_3 \bar{\Psi}_3 \quad (4.54)$$

For the middle and lower layers, the strain variation is obtained from the above equations in the usual manner.

4.2.4. Variation of stress

The equations for the stress variation are taken directly from Sect. 3.4.4 and are not modified. Naturally, the expressions for the strain and strain variation that appear in the stress variation equation are to be taken from Sects. 4.1.3 and 4.2.3

4.3. Modified Buckling Problem

There are no new concepts introduced for the modified buckling problem. The equations can be deduced from the general nonlinear equations of Sect. 3.5 by the procedure of Sect. 4.2, or the equations can be obtained from those of Sect. 4.2 by the method used in Sect. 3.5. In either case, the equations can be written directly by inspection.

4.3.1. Field equations

The field equations are Eqs. (4.39) - (4.42) of Sect. 4.2.1 and Eqs. (3.171) - (3.174) of Sect. 3.5.1. In addition, the quantities δe^{i3} now become

$$\begin{aligned}\delta \bar{e}'^{\alpha 3} &= \delta \bar{a}'^{\alpha 3}, & \delta \bar{e}^{\alpha 3} &= \delta \bar{a}^{\alpha 3}, & \delta \bar{e}''^{\alpha 3} &= \delta \bar{a}''^{\alpha 3}, \\ \delta \bar{e}'^{33} &= \delta \bar{a}'^{33} + \bar{S}'^{\gamma 3} \left[\delta \bar{U}_{3,\gamma} + \frac{\bar{h}}{2} (\delta \bar{\Psi}_{3,\gamma} - \delta \bar{\Psi}_{3,\gamma}) + \bar{\Theta}^3 \delta \bar{\Psi}_{3,\gamma} \right] + \bar{S}'^{33} \delta \bar{\Psi}_3, \\ & & & & (4.55) \\ \delta \bar{e}^{33} &= \delta \bar{a}^{33} + \bar{S}^{\gamma 3} (\delta \bar{U}_{3,\gamma} + \bar{\Theta}^3 \delta \bar{\Psi}_{3,\gamma}) + \bar{S}^{33} \delta \bar{\Psi}_3, \\ \delta \bar{e}''^{33} &= \delta \bar{a}''^{33} + \bar{S}''^{\gamma 3} \left[\delta \bar{U}_{3,\gamma} - \frac{\bar{h}}{2} (\delta \bar{\Psi}_{3,\gamma} - \delta \bar{\Psi}_{3,\gamma}) + \bar{\Theta}^3 \delta \bar{\Psi}_{3,\gamma} \right] + \bar{S}''^{33} \delta \bar{\Psi}_3.\end{aligned}$$

4.3.2. Boundary conditions

The stress boundary conditions are given by Eqs. (3.178), (3.179), (4.44) - (4.47), and (3.184) - (3.187). As before, the δe^{i3} quantities are defined by Eqs. (4.55). The displacement boundary conditions are Eqs. (3.147) - (3.149).

4.3.3. Variation of strain

It is interesting to note that the variation of strain is the same as for the general nonlinear theory, and, therefore, the equations of Sect. 3.5.3 apply directly to this case.

4.3.4. Variation of stress

The stress variation equations of Sect. 3.4.4 remain applicable to this case.

5. SIMPLIFIED LARGE DEFLECTION THEORY FOR SANDWICH PLATES WITH THIN FACINGS

In general, sandwich plates are constructed so that the upper and lower layers (i.e., the facings) are much thinner than the middle layer. For thin plates, Kirchhoff's first assumption is usually employed, which quoting from Todhunter and Pearson [24] is

Every straight line in the plate which was originally perpendicular to the plate-surfaces, remains straight after the strain and perpendicular to the surfaces which were originally planes parallel to the plate-surfaces.

Thus, for thin plates, it is assumed that the deformation due to the shear stresses is negligible, i.e.,

$$\gamma_{23} = 0. \quad (5.1)$$

Also, in thin plate theory, it is usually assumed that the distance between points on the straight lines which are normal to the plate-surfaces remain constant during deformation, or

$$\gamma_{33} = 0. \quad (5.2)$$

Novozhilov [6] has shown that with the above assumptions, i.e., Eqs. (5.1) and (5.2), and the simplified strain tensor of Sect. 4, as given by Eq. (4.1), the displacement functions u^i and ψ^i are related as follows:

$$\psi_\alpha = -u_{3,\alpha}, \quad \psi_3 = 0. \quad (5.3)$$

Then, from Eqs. (3.12),

$$\begin{aligned} \dot{\psi}_\alpha &= -\left(\bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha}\right), & \dot{\psi}_3 &= 0, \\ \ddot{\psi}_\alpha &= -\left(\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha}\right), & \ddot{\psi}_3 &= 0. \end{aligned} \quad (5.4)$$

Therefore, corresponding to Eqs. (3.12), we have

$$\begin{aligned} \dot{u}_\alpha &= \bar{u}_\alpha + \frac{\bar{h}}{2} \bar{\psi}_\alpha - \frac{\bar{h}}{2} \left(\bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha}\right), & \dot{u}_3 &= \bar{u}_3 + \frac{\bar{h}}{2} \bar{\psi}_3, \\ \ddot{u}_\alpha &= \bar{u}_\alpha - \frac{\bar{h}}{2} \bar{\psi}_\alpha + \frac{\bar{h}}{2} \left(\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha}\right), & \ddot{u}_3 &= \bar{u}_3 - \frac{\bar{h}}{2} \bar{\psi}_3; \end{aligned} \quad (5.5)$$

and corresponding to Eqs. (3.13),

$$\begin{aligned} \bar{u}_\alpha &= \frac{1}{2} \left(\dot{u}_\alpha + \ddot{u}_\alpha + \frac{\bar{h}}{2} \dot{u}_{3,\alpha} - \frac{\bar{h}}{2} \ddot{u}_{3,\alpha} \right), & \bar{u}_3 &= \frac{1}{2} \left(\dot{u}_3 + \ddot{u}_3 \right), \\ \bar{\psi}_\alpha &= \frac{1}{\bar{h}} \left(\dot{u}_\alpha - \ddot{u}_\alpha + \frac{\bar{h}}{2} \dot{u}_{3,\alpha} + \frac{\bar{h}}{2} \ddot{u}_{3,\alpha} \right), & \bar{\psi}_3 &= \frac{1}{\bar{h}} \left(\dot{u}_3 - \ddot{u}_3 \right). \end{aligned} \quad (5.6)$$

As mentioned in Sect. 3.2, the theory can be developed in terms of two sets of displacement functions, either

$$\bar{u}_i, \quad \bar{\psi}_i$$

or

$$\dot{u}_i, \quad \ddot{u}_i.$$

For consistency with the preceding work, the former set of displacement functions are chosen. Substitution of Eqs. (5.4) into Eqs. (3.14) yields the displacement expressions for the three layers as follows:

$$\begin{aligned}\dot{\bar{v}}_i &= \bar{u}_i + \frac{\bar{h}}{2} (\bar{\psi}_i + \bar{u}_{3,i} + \frac{\bar{h}}{2} \bar{\psi}_{3,i}) - \bar{\theta}^3 (\bar{u}_{3,i} + \frac{\bar{h}}{2} \bar{\psi}_{3,i}); \\ \bar{v}_i &= \bar{u}_i + \bar{\theta}^3 \bar{\psi}_i; \\ \ddot{\bar{v}}_i &= \bar{u}_i - \frac{\bar{h}}{2} (\bar{\psi}_i + \bar{u}_{3,i} - \frac{\bar{h}}{2} \bar{\psi}_{3,i}) - \bar{\theta}^3 (\bar{u}_{3,i} - \frac{\bar{h}}{2} \bar{\psi}_{3,i}).\end{aligned}\tag{5.7}$$

5.1. Fundamental Equations

In Sect. 4.1, the fundamental equations were deduced from the general equations of Sect. 3.3; obviously, a similar procedure could be employed here to reduce the equations of Sect. 4.1. to this special case. However, care must be taken if the reduction is to yield the same identical equations that would result from the direct application of the variational principle, rather than an equivalent set of equations. For example, Eqs. (4.7) and (4.8) which are

$$\begin{aligned}(\dot{M}^{\alpha\beta} - \frac{\bar{h}}{2} \dot{N}^{\alpha\beta})|_{\beta} - \dot{Q}^{\alpha} - \frac{\bar{h}}{2} \dot{p}^{\alpha} + \dot{c}^{\alpha} &= -\frac{\bar{h}}{2} \dot{a}^{\alpha} + \dot{m}^{\alpha}, \\ (\ddot{M}^{\alpha\beta} + \frac{\bar{h}}{2} \ddot{N}^{\alpha\beta})|_{\beta} - \ddot{Q}^{\alpha} + \frac{\bar{h}}{2} \ddot{p}^{\alpha} + \ddot{c}^{\alpha} &= \frac{\bar{h}}{2} \ddot{a}^{\alpha} + \ddot{m}^{\alpha}\end{aligned}\tag{5.8}$$

are obviously unaffected by the change in the form of the displacements and, therefore, should be applicable to this case; indeed, these

equations are valid. However, the origin of Eqs. (4.7) and (4.8) lies in Eq. (3.27) which shows that these equations are a direct consequence of the arbitrary functions $\delta' \psi_\alpha$ and $\delta'' \psi_\alpha$. Now, for this case,

$$\delta' \psi_\alpha = - \left(\delta \bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \delta \bar{\psi}_{3,\alpha} \right), \quad (5.9)$$

$$\delta'' \psi_\alpha = - \left(\delta \bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \delta \bar{\psi}_{3,\alpha} \right)$$

as seen from Eqs. (5.4). Obviously then, terms in the variational principle must be regrouped before applying the fundamental theorem of calculus of variation; the resulting equations are

$$\begin{aligned} & \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} - \ddot{M}^{\alpha\beta} \right] /_\beta + \dot{Q}^\alpha + \ddot{Q}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - \ddot{p}^\alpha) \\ & - \dot{c}^\alpha - \ddot{c}^\alpha = \frac{\bar{h}}{2} (\dot{a}^\alpha - \ddot{a}^\alpha) - \dot{m}^\alpha - \ddot{m}^\alpha, \end{aligned} \quad (5.10)$$

$$\begin{aligned} & \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} + \ddot{M}^{\alpha\beta} \right] /_\beta + \dot{Q}^\alpha - \ddot{Q}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha + \ddot{p}^\alpha) \\ & - \dot{c}^\alpha + \ddot{c}^\alpha = \frac{\bar{h}}{2} (\dot{a}^\alpha + \ddot{a}^\alpha) - \dot{m}^\alpha + \ddot{m}^\alpha. \end{aligned}$$

Eqs. (5.8) and (5.10) are equivalent.

As indicated above, the reduction process for this case presents no particular difficulties; however, the process is not nearly as direct as in Sect. 4.1. In fact, the beauty of the variational principle becomes obscured by the reduction process; therefore, the equations for this section will be obtained by the direct application of the variational principle.

The modified Hellinger-Reissner equation for the simplified strain tensor has already been developed, Eq. (4.2); therefore, the fundamental equations are a result of the following equations:

$$\int_V \left\{ [S^{ij}(\delta_i^r + \delta_3^r v_{3,i})] |_{,j} + \rho_0 ({}_0F^r - {}_0f^r) \right\} \delta v_r dV = 0; \quad (5.11a)$$

$$\int_{A_s} [\tilde{S}_*^i - m_j S^{jr} (\delta_r^i + \delta_{3,r}^i)] \delta v_i dA = 0; \quad (5.11b)$$

$$\int_{A_v} (v_i - \tilde{v}_i) \delta S_*^i dA = 0; \quad (5.11c)$$

$$\int_V \left[\gamma_{ij} - \frac{1}{2} (v_{i,j} + v_{j,i} + v_{3,i} v_{3,j}) \right] \delta S^{ij} dV = 0; \quad (5.11d)$$

$$\int_V \left[S^{ij} - \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} - \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right) \right] \delta \gamma_{ij} dV = 0. \quad (5.11e)$$

5.1.1. Equations of motion

The procedure that will be used here is in general the same as that employed in Sect. 3.3.1. Consider Eq. (5.11a), i.e.,

$$\int_V \left\{ [S^{ij}(\delta_i^r + \delta_3^r v_{3,i})] |_{,j} + \rho_0 ({}_0F^r - {}_0f^r) \right\} \delta v_r dV = 0,$$

which in terms of subtensors becomes

$$\begin{aligned} \int_V \left\{ [S^{\alpha\beta} |_{,\beta} + S^{\alpha 3}_{,3} + \rho_0 ({}_0F^\alpha - {}_0f^\alpha)] \delta v_\alpha \right. \\ \left. + [A^{3\beta} |_{,\beta} + B^{33}_{,3} + \rho_0 ({}_0F^3 - {}_0f^3)] \delta v_3 \right\} dV = 0 \end{aligned} \quad (5.12)$$

where

$$\begin{aligned} A^{3\beta} &= S^{\alpha\beta} v_{3,\alpha} + S^{\beta 3} (1 + v_{3,3}) , \\ B^{33} &= S^{\alpha 3} v_{3,\alpha} + S^{33} (1 + v_{3,3}) . \end{aligned} \quad (5.13)$$

In terms of the three layers, Eq. (5.12) becomes

$$\begin{aligned} & \int_A \left\{ \int_b^a \left[(\dot{S}^{\alpha\beta}/\beta + \dots) \delta \dot{v}_\alpha + (\dot{A}^{3\beta}/\beta + \dots) \delta \dot{v}_3 \right] \right. \\ & + \int_c^b \left[(\bar{S}^{\alpha\beta}/\beta + \dots) \delta \bar{v}_\alpha + (\bar{A}^{3\beta}/\beta + \dots) \delta \bar{v}_3 \right] \\ & \left. + \int_d^c \left[(\ddot{S}^{\alpha\beta}/\beta + \dots) \delta \ddot{v}_\alpha + (\ddot{A}^{3\beta}/\beta + \dots) \delta \ddot{v}_3 \right] \right\} d\bar{\theta}^3 dA = 0. \end{aligned} \quad (5.14)$$

Next, Eqs. (5.7) are substituted into the above equation to obtain

$$\begin{aligned}
& \left[\left\{ \int_b^a \left[\dot{S}^{\alpha\beta}/\rho + \dot{S}^{\alpha 3}_{,3} + \dot{\rho}_o ({}_oF^\alpha - {}_of^\alpha) \right] \right. \right. \\
& + \int_c^b \left[\bar{S}^{\alpha\beta}/\rho + \dots \right] + \int_d^c \left[\ddot{S}^{\alpha\beta}/\rho + \dots \right] \left. \right\} \delta \bar{u}_\alpha \\
& + \left\{ \int_b^a \frac{\bar{h}}{2} \left[\dot{S}^{\alpha\beta}/\rho + \dots \right] + \int_c^b \bar{\theta}^3 \left[\bar{S}^{\alpha\beta}/\rho + \dots \right] \right. \\
& - \left. \int_d^c \frac{\bar{h}}{2} \left[\ddot{S}^{\alpha\beta}/\rho + \dots \right] \right\} \delta \bar{\psi}_\alpha \\
& - \left\{ \int_b^a \left(\bar{\theta}^3 - \frac{\bar{h}}{2} \right) \left[\dot{S}^{\alpha\beta}/\rho + \dots \right] \right. \\
& + \left. \int_d^c \left(\bar{\theta}^3 + \frac{\bar{h}}{2} \right) \left[\ddot{S}^{\alpha\beta}/\rho + \dots \right] \right\} \delta \bar{u}_{3,\alpha} \\
& - \left\{ \int_b^a \frac{\bar{h}}{2} \left(\bar{\theta}^3 - \frac{\bar{h}}{2} \right) \left[\dot{S}^{\alpha\beta}/\rho + \dots \right] \right. \\
& - \left. \int_d^c \frac{\bar{h}}{2} \left(\bar{\theta}^3 + \frac{\bar{h}}{2} \right) \left[\ddot{S}^{\alpha\beta}/\rho + \dots \right] \right\} \delta \bar{\psi}_{3,\alpha} \\
& + \left\{ \int_b^a \left[\dot{D}^{3\beta}/\rho + \dot{E}^{33}_{,3} + \dot{\rho}_o ({}_oF^3 - {}_of^3) \right] \right. \\
& + \int_c^b \left[\bar{D}^{3\beta}/\rho + \dots \right] + \int_d^c \left[\ddot{D}^{3\beta}/\rho + \dots \right] \left. \right\} \delta \bar{u}_3 \\
& + \left\{ \int_b^a \frac{\bar{h}}{2} \left[\dot{D}^{3\beta}/\rho + \dots \right] + \int_c^b \bar{\theta}^3 \left[\bar{D}^{3\beta}/\rho + \dots \right] \right. \\
& - \left. \int_d^c \frac{\bar{h}}{2} \left[\ddot{D}^{3\beta}/\rho + \dots \right] \right\} \delta \bar{\psi}_3 \left. \right] d\bar{\theta}^3 dA = 0.
\end{aligned}$$

where

$$\begin{aligned}
 \dot{D}^{3\beta} &= \dot{S}^{\alpha\beta} \left(\bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + \dot{S}^{\beta 3}, \\
 \bar{D}^{3\beta} &= \bar{S}^{\alpha\beta} \left(\bar{u}_{3,\alpha} + \bar{\theta}^3 \bar{\psi}_{3,\alpha} \right) + \bar{S}^{\beta 3} (1 + \bar{\psi}_3), \\
 {}''D^{3\beta} &= {}''S^{\alpha\beta} \left(\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + {}''S^{\beta 3}, \\
 \dot{E}^{33} &= \dot{S}^{\alpha 3} \left(\bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + \dot{S}^{33}, \\
 \bar{E}^{33} &= \bar{S}^{\alpha 3} \left(\bar{u}_{3,\alpha} + \bar{\theta}^3 \bar{\psi}_{3,\alpha} \right) + \bar{S}^{33} (1 + \bar{\psi}_3), \\
 {}''E^{33} &= {}''S^{\alpha 3} \left(\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + {}''S^{33}.
 \end{aligned} \tag{5.16}$$

By using equations corresponding to Eqs. (3.26), and by using the equations for the stress resultants Eqs. (3.28) - (3.34), the fundamental theorem of calculus of variations can be applied to Eq. (5.15) to obtain the following equations of motion:

$$\begin{aligned}
 (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + {}''N^{\alpha\beta})|_{\beta} + \dot{p}^{\alpha} + \bar{p}^{\alpha} + {}''p^{\alpha} &= \dot{a}^{\alpha} + \bar{a}^{\alpha} + {}''a^{\alpha}; \\
 \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - {}''N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right]|_{\beta} - \bar{Q}^{\alpha} + \frac{\bar{h}}{2} (\dot{p}^{\alpha} - {}''p^{\alpha}) + \bar{c}^{\alpha} \\
 &= \frac{\bar{h}}{2} (\dot{a}^{\alpha} - {}''a^{\alpha}) + \bar{m}^{\alpha};
 \end{aligned} \tag{5.17}$$

$$\tag{5.18}$$

$$\left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} - \ddot{M}^{\alpha\beta} \right] /_{\beta} + \dot{Q}^{\alpha} + \ddot{Q}^{\alpha} + \frac{\bar{h}}{2} (\dot{p}^{\alpha} - \ddot{p}^{\alpha}) - \dot{c}^{\alpha} - \ddot{c}^{\alpha} = \frac{\bar{h}}{2} (\dot{a}^{\alpha} - \ddot{a}^{\alpha}) - \dot{m}^{\alpha} - \ddot{m}^{\alpha}; \quad (5.19)$$

$$\left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} + \ddot{M}^{\alpha\beta} \right] /_{\beta} + \dot{Q}^{\alpha} - \ddot{Q}^{\alpha} + \frac{\bar{h}}{2} (\dot{p}^{\alpha} + \ddot{p}^{\alpha}) - \dot{c}^{\alpha} + \ddot{c}^{\alpha} = \frac{\bar{h}}{2} (\dot{a}^{\alpha} + \ddot{a}^{\alpha}) - \dot{m}^{\alpha} + \ddot{m}^{\alpha}; \quad (5.20)$$

$$\left\{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{u}_{3,\gamma} + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + \dot{Q}^{\beta} + \bar{Q}^{\beta} (1 + \bar{\psi}_3) + \ddot{Q}^{\beta} \right\} /_{\beta} + \dot{p}^3 + \bar{p}^3 + \ddot{p}^3 = \dot{a}^3 + \bar{a}^3 + \ddot{a}^3; \quad (5.21)$$

$$\left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} + \frac{\bar{h}}{2} (\dot{Q}^{\beta} - \ddot{Q}^{\beta}) \right\} /_{\beta} - \bar{Q}^{\alpha} \bar{u}_{3,\alpha} + (\bar{T}^{\beta} /_{\beta} - \bar{N}^{33}) (1 + \bar{\psi}_3) + \frac{\bar{h}}{2} (\dot{p}^3 - \ddot{p}^3) + \bar{c}^3 = \frac{\bar{h}}{2} (\dot{a}^3 - \ddot{a}^3) + \bar{m}^3. \quad (5.22)$$

In the above equations, the following definitions have been used for

p^i and c^i ,

$$\left\{ \begin{matrix} m \dot{p}^{\alpha}, & m \dot{c}^{\alpha} \\ m \bar{p}^{\alpha}, & m \bar{c}^{\alpha} \\ m \ddot{p}^{\alpha}, & m \ddot{c}^{\alpha} \end{matrix} \right\} = \left\{ \begin{matrix} \dot{S}^{\alpha 3}, & \bar{\theta}^3 \dot{S}^{\alpha 3} \\ \bar{S}^{\alpha 3}, & \bar{\theta}^3 \bar{S}^{\alpha 3} \\ \ddot{S}^{\alpha 3}, & \bar{\theta}^3 \ddot{S}^{\alpha 3} \end{matrix} \right\}, \quad AT \left\{ \begin{matrix} \dot{\theta}^3 \\ \bar{\theta}^3 \\ \ddot{\theta}^3 \end{matrix} \right\} = (-1)^m \left\{ \begin{matrix} \frac{\dot{h}}{2} \\ \bar{\frac{h}{2}} \\ \ddot{\frac{h}{2}} \end{matrix} \right\}, \quad (5.23)$$

$$\begin{Bmatrix} m' \bar{p}^3, m' \bar{c}^3 \\ m \bar{p}^3, m \bar{c}^3 \\ m'' \bar{p}^3, m'' \bar{c}^3 \end{Bmatrix} = \begin{Bmatrix} \bar{E}^{33}, \bar{\theta}^3 \bar{E}^{33} \\ \bar{E}^{33}, \bar{\theta}^3 \bar{E}^{33} \\ \bar{E}^{33}, \bar{\theta}^3 \bar{E}^{33} \end{Bmatrix}, \quad AT \begin{Bmatrix} \bar{\theta}^3 \\ \bar{\theta}^3 \\ \bar{\theta}^3 \end{Bmatrix} = (-1)^m \begin{Bmatrix} \frac{h}{2} \\ \frac{h}{2} \\ \frac{h}{2} \end{Bmatrix}, \quad (5.24)$$

along with Eq. (3.34).

It should be mentioned that when the fundamental theorem of calculus of variations was applied to Eq. (5.15), the quantities $\delta \bar{u}_{3,\alpha}$ and $\delta \bar{u}_3$ (also, $\delta \bar{\psi}_{3,\alpha}$ and $\delta \bar{\psi}_3$) were assumed to be arbitrary independent functions. Rather than use this approach, one could use the relation

$$\begin{aligned} \int_V (\dots) \delta \bar{u}_{3,\alpha} dV &= \int_A (\dots) \delta \bar{u}_3 m_\alpha dA \\ &- \int_V (\dots) /_\alpha \delta \bar{u}_3 dV \end{aligned} \quad (5.25)$$

to combine terms; this has two effects, namely, to decrease the number of the equations of motion by increasing their order, and to add additional terms to the stress boundary conditions. Naturally, the equations are equivalent, regardless of which method is used. In fact, the equations of motion that would result by using the above relation (and the corresponding relation between $\delta \bar{\psi}_{3,\alpha}$ and $\delta \bar{\psi}_3$) can be obtained from Eqs. (5.17) - (5.21) by merely substituting Eqs. (5.19) and (5.20) into Eqs. (5.21) and (5.22), respectively, through the

stress resultants $'Q^\alpha$ and $''Q^\alpha$. Thus, the alternate set of equations of motion is

$$(\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + ''N^{\alpha\beta})|_\beta + \dot{p}^\alpha + \bar{p}^\alpha + ''p^\alpha = \dot{a}^\alpha + \bar{a}^\alpha + ''a^\alpha, \quad (5.26)$$

$$\begin{aligned} & \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - ''N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] |_\beta - \bar{Q}^\alpha + \frac{\bar{h}}{2} (\dot{p}^\alpha - ''p^\alpha) + \bar{c}^\alpha \\ &= \frac{\bar{h}}{2} (\dot{a}^\alpha - ''a^\alpha) + \bar{m}^\alpha, \end{aligned} \quad (5.27)$$

$$\begin{aligned} & \{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + ''N^{\beta\gamma}) \bar{u}_{3,\gamma} + \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - ''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} \\ &+ \frac{\bar{h}}{2} (-\dot{p}^\beta + ''p^\beta + \dot{a}^\beta - ''a^\beta) + \dot{c}^\beta + \bar{c}^\beta - \dot{m}^\beta - ''m^\beta \\ &+ \bar{Q}^\beta (1 + \bar{\psi}_3) \} |_\beta - \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - ''N^{\beta\gamma}) - \dot{M}^{\beta\gamma} - ''M^{\beta\gamma} \right] |_{\beta\gamma} \\ &+ \dot{p}^3 + \bar{p}^3 + ''p^3 = \dot{a}^3 + \bar{a}^3 + ''a^3, \end{aligned} \quad (5.28)$$

$$\begin{aligned} & \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - ''N^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + ''N^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} \right. \\ &- \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{p}^\beta + ''p^\beta - \dot{a}^\beta - ''a^\beta) - \dot{c}^\beta + \bar{c}^\beta + \dot{m}^\beta - ''m^\beta \right] \} |_\beta \\ &- \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} + ''N^{\beta\gamma}) - \dot{M}^{\beta\gamma} + ''M^{\beta\gamma} \right] |_{\beta\gamma} + (\bar{T}^\beta |_\beta - \bar{N}^{33}) (1 + \bar{\psi}_3) \\ &- \bar{Q}^\alpha \bar{u}_{3,\alpha} + \frac{\bar{h}}{2} (\dot{p}^3 - ''p^3) + \bar{c}^3 = \frac{\bar{h}}{2} (\dot{a}^3 - ''a^3) + \bar{m}^3. \end{aligned} \quad (5.29)$$

5.1.2. Boundary conditions

Eqs. (5.11b) and (5.11c) are used to establish the boundary conditions. First, for the stress boundary conditions, consider Eq. (5.11b), i.e.,

$$\int_{A_s} [\tilde{S}_*^i - n_j S^{jr} (\delta_r^i + \delta_3^i v_{3,r})] \delta v_i dA = 0.$$

By the procedure of Sect. 3.3.2, the equation corresponding to Eq. (3.48) is

$$\begin{aligned} & \int_{A_s} \left\{ (\tilde{S}_*^{\alpha} - S^{\alpha 3}) \left[\delta \bar{u}_{\alpha} + \frac{\bar{h}}{2} \delta \bar{\psi}_{\alpha} - (\bar{\theta}^3 - \frac{\bar{h}}{2}) \delta \bar{u}_{3,\alpha} - \frac{\bar{h}}{2} (\bar{\theta}^3 - \frac{\bar{h}}{2}) \delta \bar{\psi}_{3,\alpha} \right] \right. \\ & \quad \left. + (\tilde{S}_*^3 - E^{33}) (\delta \bar{u}_3 + \frac{\bar{h}}{2} \delta \bar{\psi}_3) \right\} dA \\ & + \int_{A_s} \left\{ (\tilde{S}_*^{\alpha} + S^{\alpha 3}) \left[\delta \bar{u}_{\alpha} - \frac{\bar{h}}{2} \delta \bar{\psi}_{\alpha} - (\bar{\theta}^3 + \frac{\bar{h}}{2}) \delta \bar{u}_{3,\alpha} + \frac{\bar{h}}{2} (\bar{\theta}^3 + \frac{\bar{h}}{2}) \delta \bar{\psi}_{3,\alpha} \right] \right. \\ & \quad \left. + (\tilde{S}_*^3 + E^{33}) (\delta \bar{u}_3 - \frac{\bar{h}}{2} \delta \bar{\psi}_3) \right\} dA \\ & + \int_{A_s} \left\{ \left[\int_b^a (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) + \int_c^b (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} \bar{S}^{\alpha \beta}) + \int_d^c (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) \right] \delta \bar{u}_{\alpha} \right. \\ & \quad + \left[\int_b^a \frac{\bar{h}}{2} (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) + \int_c^b \bar{\theta}^3 (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} \bar{S}^{\alpha \beta}) - \int_d^c \frac{\bar{h}}{2} (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) \right] \delta \bar{\psi}_{\alpha} \\ & \quad - \left[\int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) + \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) \right] \delta \bar{u}_{3,\alpha} \\ & \quad - \frac{\bar{h}}{2} \left[\int_b^a (\bar{\theta}^3 - \frac{\bar{h}}{2}) (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) + \int_d^c (\bar{\theta}^3 + \frac{\bar{h}}{2}) (\tilde{S}_*^{\alpha} - \bar{m}_{\beta} S^{\alpha \beta}) \right] \delta \bar{\psi}_{3,\alpha} \\ & \quad + \left[\int_b^a (\tilde{S}_*^3 - \bar{m}_{\beta} D^{3\beta}) + \int_c^b (\tilde{S}_*^3 - \bar{m}_{\beta} \bar{D}^{3\beta}) + \int_d^c (\tilde{S}_*^3 - \bar{m}_{\beta} D^{3\beta}) \right] \delta \bar{u}_3 \\ & \quad \left. + \left[\int_b^a \frac{\bar{h}}{2} (\tilde{S}_*^3 - \bar{m}_{\beta} D^{3\beta}) + \int_c^b \bar{\theta}^3 (\tilde{S}_*^3 - \bar{m}_{\beta} \bar{D}^{3\beta}) - \int_d^c \frac{\bar{h}}{2} (\tilde{S}_*^3 - \bar{m}_{\beta} D^{3\beta}) \right] \delta \bar{\psi}_3 \right\} dA = 0. \end{aligned}$$

Then, the application of the fundamental theorem of calculus of variation yields the stress boundary conditions as follows:

on the top surface,

$${}_2p_*^i - {}_2p^i = 0, \quad {}_2c_*^i - {}_2c^i = 0; \quad (5.31)$$

on the bottom surface,

$${}_1p_*^i + {}_1p^i = 0, \quad {}_1c_*^i + {}_1c^i = 0; \quad (5.32)$$

on the edge surface,

$${}_0s' + {}_0\bar{s} + {}_0s'' = {}_0\bar{m}_\beta (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + N''^{\alpha\beta}), \quad (5.33)$$

$$\frac{\bar{h}}{2}({}_0s' - {}_0s'') + {}_0\bar{\ell} = {}_0\bar{m}_\beta \left[\frac{\bar{h}}{2}(\dot{N}^{\alpha\beta} - N''^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right], \quad (5.34)$$

$${}_0\bar{\ell}' + {}_0\bar{\ell}'' - \frac{\bar{h}}{2}({}_0s' - {}_0s'') = {}_0\bar{m}_\beta \left[\dot{M}^{\alpha\beta} + \bar{M}^{\alpha\beta} - \frac{\bar{h}}{2}(\dot{N}^{\alpha\beta} - N''^{\alpha\beta}) \right], \quad (5.35)$$

$${}_0\bar{\ell}' - {}_0\bar{\ell}'' - \frac{\bar{h}}{2}({}_0s' + {}_0s'') = {}_0\bar{m}_\beta \left[\dot{M}^{\alpha\beta} - \bar{M}^{\alpha\beta} - \frac{\bar{h}}{2}(\dot{N}^{\alpha\beta} + N''^{\alpha\beta}) \right], \quad (5.36)$$

$$\begin{aligned} {}_0s^3 + {}_0\bar{s}^3 + {}_0s^3 = {}_0\bar{m}_\beta \{ & (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + N''^{\alpha\beta}) \bar{u}_{3,\alpha} + \dot{Q}^\beta + \bar{Q}^\beta (1 + \bar{\psi}_3) \\ & + \bar{Q}''^\beta + \left[\frac{\bar{h}}{2}(\dot{N}^{\alpha\beta} - N''^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{\psi}_{3,\alpha} \}, \end{aligned} \quad (5.37)$$

$$\begin{aligned} \frac{\bar{h}}{2}({}_0s^3 - {}_0s^3) + {}_0\bar{\ell}^3 = {}_0\bar{m}_\beta \{ & \left[\frac{\bar{h}}{2}(\dot{N}^{\alpha\beta} - N''^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{u}_{3,\alpha} + \frac{\bar{h}}{2}(\dot{Q}^\beta - \bar{Q}''^\beta) \\ & + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\alpha\beta} + N''^{\alpha\beta}) + \bar{K}^{\alpha\beta} \right] \bar{\psi}_{3,\alpha} + \bar{T}^\beta (1 + \bar{\psi}_3) \}, \end{aligned} \quad (5.38)$$

where Eqs. (3.28) - (3.30) and (3.49) have been applied. These stress boundary conditions are associated with the equations of motion given by Eqs. (5.17) - (5.22).

For the other set of equations of motion, Eqs. (5.26) - (5.28), a corresponding set of stress boundary conditions must be developed, as discussed in Sect. 5.11. By substituting the quantities

$$\dot{Q}^\alpha + \ddot{Q}^\alpha \quad \text{and} \quad \dot{Q}^\alpha - \ddot{Q}^\alpha,$$

obtained from Eqs. (5.19) and (5.20), into Eqs. (5.37) and (5.38), the stress boundary conditions are Eqs. (5.31) - (5.36) and

$$\begin{aligned} \dot{S}_o^3 + \ddot{S}_o^3 + \ddot{S}_o^3 = {}_o\bar{m}_\beta \left\{ (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) \bar{u}_{3,\alpha} + \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{\psi}_{3,\alpha} \right. \\ \left. - \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} - \ddot{M}^{\alpha\beta} \right] \right|_\alpha - \frac{\bar{h}}{2} (\dot{p}^\beta - \ddot{p}^\beta - \dot{a}^\beta + \ddot{a}^\beta) \\ \left. + \dot{c}^\beta + \ddot{c}^\beta - \dot{m}^\beta - \ddot{m}^\beta + \bar{Q}^\beta (1 + \bar{\psi}_3) \right\}, \end{aligned} \quad (5.39)$$

$$\begin{aligned} \frac{\bar{h}}{2} (\dot{S}_o^3 - \ddot{S}_o^3) + {}_o\bar{\ell}^3 = {}_o\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{u}_{3,\alpha} + \bar{T}^\beta (1 + \bar{\psi}_3) \right. \\ \left. + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) + \bar{K}^{\alpha\beta} \right] \bar{\psi}_{3,\alpha} - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) - \dot{M}^{\alpha\beta} + \ddot{M}^{\alpha\beta} \right] \right|_\alpha \\ \left. - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{p}^\beta + \ddot{p}^\beta - \dot{a}^\beta - \ddot{a}^\beta) - \dot{c}^\beta + \ddot{c}^\beta + \dot{m}^\beta - \ddot{m}^\beta \right] \right\}. \end{aligned} \quad (5.40)$$

The displacement boundary conditions are the same as in Sect. 3.3.2 and are given by Eqs. (3.61) - (3.63).

5.1.3. Strain-displacement equations

From Eq. (5.1ld), the strain-displacement relations are a result of the equation

$$\int_V \left[\gamma_{ij} - \frac{1}{2} (\dot{v}_{i/j} + \dot{v}_{j/i} + \dot{v}_{3,i} \dot{v}_{3,j}) \right] \delta S^{ij} dV = 0.$$

With the basic premise of this section, Eqs. (5.1) and (5.2) for the facings, the above equation becomes

$$\begin{aligned} & \int_V \left[\dot{\gamma}_{\alpha\beta} - \frac{1}{2} (\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}_{3,\alpha} \dot{v}_{3,\beta}) \right] \delta \dot{S}^{\alpha\beta} dV \\ & + \int_{\bar{V}} \left[\bar{\gamma}_{ij} - \frac{1}{2} (\bar{v}_{i/j} + \bar{v}_{j/i} + \bar{v}_{3,i} \bar{v}_{3,j}) \right] \delta \bar{S}^{ij} dV \quad (5.41) \\ & + \int_V \left[\ddot{\gamma}_{\alpha\beta} - \frac{1}{2} (\ddot{v}_{\alpha/\beta} + \ddot{v}_{\beta/\alpha} + \ddot{v}_{3,\alpha} \ddot{v}_{3,\beta}) \right] \delta \ddot{S}^{\alpha\beta} dV = 0 \end{aligned}$$

when written in terms of all three layers. From the fundamental theorem of calculus of variations, after the middle layer has been expressed in subtensor form, Eq. (5.41) yields

$$\begin{aligned} \dot{\gamma}_{\alpha\beta} &= \frac{1}{2} (\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}_{3,\alpha} \dot{v}_{3,\beta}), \\ \bar{\gamma}_{\alpha\beta} &= \frac{1}{2} (\bar{v}_{\alpha/\beta} + \bar{v}_{\beta/\alpha} + \bar{v}_{3,\alpha} \bar{v}_{3,\beta}), \\ \bar{\gamma}_{\alpha 3} &= \frac{1}{2} (\bar{v}_{\alpha,3} + \bar{v}_{3,\alpha} + \bar{v}_{3,\alpha} \bar{v}_{3,3}), \\ \bar{\gamma}_{33} &= \bar{v}_{3,3} + \frac{1}{2} \bar{v}_{3,3} \bar{v}_{3,3}, \\ \ddot{\gamma}_{\alpha\beta} &= \frac{1}{2} (\ddot{v}_{\alpha/\beta} + \ddot{v}_{\beta/\alpha} + \ddot{v}_{3,\alpha} \ddot{v}_{3,\beta}). \end{aligned} \quad (5.42)$$

After substituting the expressions for the displacements Eqs. (5.7) into the above, the strain-displacement equations become

$$\begin{aligned} \gamma_{\alpha\beta} = \frac{1}{2} & \left[\bar{u}_{\alpha/\beta} + \bar{u}_{\beta/\alpha} + \bar{u}_{3,\alpha} \bar{u}_{3,\beta} + \frac{\bar{h}}{2} (\bar{\psi}_{\alpha/\beta} + \bar{\psi}_{\beta/\alpha} \right. \\ & + 2 \bar{u}_{3,\alpha\beta} + \bar{u}_{3,\alpha} \bar{\psi}_{3,\beta} + \bar{\psi}_{3,\alpha} \bar{u}_{3,\beta}) \\ & + \left(\frac{\bar{h}}{2} \right)^2 (2 \bar{\psi}_{3,\alpha\beta} + \bar{\psi}_{3,\alpha} \bar{\psi}_{3,\beta}) - 2 \bar{\theta}^3 (\bar{u}_{3,\alpha\beta} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha\beta}) \Big], \end{aligned} \quad (5.43)$$

$$\begin{aligned} \bar{\gamma}_{\alpha\beta} = \frac{1}{2} & \left[\bar{u}_{\alpha/\beta} + \bar{u}_{\beta/\alpha} + \bar{u}_{3,\alpha} \bar{u}_{3,\beta} + \bar{\theta}^3 (\bar{\psi}_{\alpha/\beta} + \bar{\psi}_{\beta/\alpha} \right. \\ & + \bar{u}_{3,\alpha} \bar{\psi}_{3,\beta} + \bar{\psi}_{3,\alpha} \bar{u}_{3,\beta}) + (\bar{\theta}^3)^2 \bar{\psi}_{3,\alpha} \bar{\psi}_{3,\beta} \Big], \end{aligned} \quad (5.44)$$

$$\bar{\gamma}_{\alpha 3} = \frac{1}{2} \left[\bar{\psi}_{\alpha} + \bar{u}_{3,\alpha} + \bar{\psi}_3 \bar{u}_{3,\alpha} + \bar{\theta}^3 (\bar{\psi}_{3,\alpha} + \bar{\psi}_3 \bar{\psi}_{3,\alpha}) \right], \quad (5.45)$$

$$\bar{\gamma}_{33} = \bar{\psi}_3 + \frac{1}{2} \bar{\psi}_3 \bar{\psi}_3 \quad (5.46)$$

$$\begin{aligned} \gamma''_{\alpha\beta} = \frac{1}{2} & \left[\bar{u}_{\alpha/\beta} + \bar{u}_{\beta/\alpha} + \bar{u}_{3,\alpha} \bar{u}_{3,\beta} - \frac{\bar{h}}{2} (\bar{\psi}_{\alpha/\beta} + \bar{\psi}_{\beta/\alpha} \right. \\ & + 2 \bar{u}_{3,\alpha\beta} + \bar{u}_{3,\alpha} \bar{\psi}_{3,\beta} + \bar{\psi}_{3,\alpha} \bar{u}_{3,\beta}) \\ & + \left(\frac{\bar{h}}{2} \right)^2 (2 \bar{\psi}_{3,\alpha\beta} + \bar{\psi}_{3,\alpha} \bar{\psi}_{3,\beta}) - 2 \bar{\theta}^3 (\bar{u}_{3,\alpha\beta} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha\beta}) \Big]. \end{aligned} \quad (5.47)$$

5.1.4. Stress-strain equations

The stress-strain equations are obtained from Eq. (5.11e), i.e.,

$$\int_V \left[S^{ij} - \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{ij}} + \frac{\partial \Sigma^*}{\partial \gamma_{ji}} \right) \right] \delta \gamma_{ij} dV = 0.$$

With the premise of

$$\dot{\gamma}_{i3} = \ddot{\gamma}_{i3} = 0,$$

Eq. (5.11e) becomes

$$\begin{aligned} & \int_V \left[\dot{S}^{\alpha\beta} - \frac{1}{2} \left(\frac{\partial \dot{\Sigma}^*}{\partial \dot{\gamma}_{\alpha\beta}} + \frac{\partial \dot{\Sigma}^*}{\partial \dot{\gamma}_{\beta\alpha}} \right) \right] \delta \dot{\gamma}_{\alpha\beta} dV \\ & + \int_V \left[\bar{S}^{ij} - \frac{1}{2} \left(\frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{ij}} + \frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{ji}} \right) \right] \delta \bar{\gamma}_{ij} dV \\ & \int_V \left[\ddot{S}^{\alpha\beta} - \frac{1}{2} \left(\frac{\partial \ddot{\Sigma}^*}{\partial \ddot{\gamma}_{\alpha\beta}} + \frac{\partial \ddot{\Sigma}^*}{\partial \ddot{\gamma}_{\beta\alpha}} \right) \right] \delta \ddot{\gamma}_{\alpha\beta} dV = 0 \end{aligned} \quad (5.48)$$

when written in terms of the three layers. Application of the fundamental theorem of calculus of variations gives

$$\begin{aligned} \dot{S}^{\alpha\beta} &= \frac{1}{2} \left(\frac{\partial \dot{\Sigma}^*}{\partial \dot{\gamma}_{\alpha\beta}} + \frac{\partial \dot{\Sigma}^*}{\partial \dot{\gamma}_{\beta\alpha}} \right), \\ \bar{S}^{\alpha\beta} &= \frac{1}{2} \left(\frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{\alpha\beta}} + \frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{\beta\alpha}} \right), \\ \ddot{S}^{\alpha 3} &= \frac{1}{2} \left(\frac{\partial \ddot{\Sigma}^*}{\partial \ddot{\gamma}_{\alpha 3}} + \frac{\partial \ddot{\Sigma}^*}{\partial \ddot{\gamma}_{3\alpha}} \right), \end{aligned} \quad (5.49a)$$

$$\bar{S}^{33} = \frac{\partial \bar{\Sigma}^*}{\partial \bar{\gamma}_{33}}, \quad (5.49b)$$

$$S^{\alpha\beta} = \frac{1}{2} \left(\frac{\partial \Sigma^*}{\partial \gamma_{\alpha\beta}} + \frac{\partial \Sigma^{\alpha\beta}}{\partial \gamma_{\beta\alpha}} \right).$$

Here we come to the well-known but interesting paradox of thin plate theory. Note that the variational principle does not yield expressions for $'S^{i3}$ or $''S^{i3}$, which implies that they are negligible. However, the quantities $'Q^\alpha$ and $''Q^\alpha$, which are defined in terms of $'S^{\alpha 3}$ and $''S^{\alpha 3}$, respectively, appear in the equations of motion, Eqs. (5.19) and (5.20), and the stress boundary conditions, Eqs. (5.37) and (5.38). In addition, the general stress-strain equations of Sect. 3.3.4, in particular Eq. (3.107), show that the condition

$$'S^{33} = ''S^{33} = 0$$

is incompatible with the basic assumption of the present theory, namely,

$$' \gamma_{33} = '' \gamma_{33} = 0.$$

A complete discussion of this paradox for thin plate (and/or shell) theory is given by Gol'denveizer [25]. Here, we merely observe that above irregularities are a natural development from the variational principle and are in complete agreement with the fundamental hypotheses of thin plate theory. Also, it is now apparent that of the two sets of equations of motion (and corresponding stress boundary conditions)

the higher order equations are preferable because the quantities $'Q^\alpha$ and $''Q^\alpha$ do not appear.

Eqs. (5.49) give the stress in terms of the strain energy density. For the deformation theory of plasticity, the strain energy density was established in Sect. 3.3.4. for a three-dimensional continuous medium. Therefore, the results of Sect. 3.3.4 can be applied to the present theory. For example, one form for the strain energy density is given by Eq. (3.98), which in terms of subtensors becomes

$$\begin{aligned} \Sigma^* = & \frac{S_0}{\gamma_0} (\gamma_\beta^\alpha \gamma_\alpha^\beta + 2 \gamma_3^\alpha \gamma_\alpha^3 + \gamma_3^3 \gamma_3^3) \\ & + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) (\gamma_\alpha^\alpha \gamma_\beta^\beta + 2 \gamma_\alpha^\alpha \gamma_3^3 + \gamma_3^3 \gamma_3^3). \end{aligned} \quad (5.50)$$

Partial differentiation of Eq. (5.50) gives

$$\begin{aligned} \frac{\partial \Sigma^*}{\partial \gamma_\beta^\alpha} = & \frac{S_0}{\gamma_0} (\gamma_\beta^\alpha g^{\beta\gamma} \delta_\alpha^\gamma + \gamma_\alpha^\beta g^{\alpha\gamma} \delta_\beta^\gamma) \\ & + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) (2 \gamma_\alpha^\alpha g^{\beta\gamma} \delta_\beta^\gamma + 2 \gamma_3^3 g^{\alpha\gamma} \delta_\alpha^\gamma), \\ \frac{\partial \Sigma^*}{\partial \gamma_{3\gamma}^\alpha} = & \frac{S_0}{\gamma_0} (2 \gamma_3^\alpha g^{33} \delta_\alpha^\gamma + 2 \gamma_\alpha^3 g^{\alpha 3} \delta_3^\gamma), \\ \frac{\partial \Sigma^*}{\partial \gamma_{33}} = & \frac{S_0}{\gamma_0} (2 \gamma_3^3 g^{33} \delta_3^3) + \frac{1}{6} (3k - 2 \frac{S_0}{\gamma_0}) (2 \gamma_\alpha^\alpha + 2 \gamma_3^3) g^{33} \delta_3^3. \end{aligned} \quad (5.51)$$

Because of the symmetry of γ_{ij} and Eqs. (5.51), the stress-strain relations for the middle layer are

$$\begin{aligned}
\bar{S}^{\alpha\beta} &= 2 \frac{\bar{S}_0}{\bar{Y}_0} \bar{\gamma}^{\alpha\beta} + \frac{1}{3} \left(3\bar{k} - 2 \frac{\bar{S}_0}{\bar{Y}_0} \right) g^{\alpha\beta} \bar{\gamma}_k^k \\
\bar{S}^{\alpha 3} &= 2 \frac{\bar{S}_0}{\bar{Y}_0} \bar{\gamma}^{\alpha 3} \\
\bar{S}^{33} &= 2 \frac{\bar{S}_0}{\bar{Y}_0} \bar{\gamma}_3^3 + \frac{1}{3} \left(3\bar{k} - 2 \frac{\bar{S}_0}{\bar{Y}_0} \right) \bar{\gamma}_k^k
\end{aligned} \tag{5.52}$$

where Eqs. (5.51) have been substituted into Eqs. (5.49). Although in a different form, the above equations are the same as Eqs. (3.107). For the upper and lower layers, a more subtle approach must be taken because of the inconsistencies of thin plate theory, as previously mentioned. Eqs. (5.49) and (5.51) together yield

$$S^{\alpha\beta} = 2 \frac{S_0}{Y_0} \gamma^{\alpha\beta} + \frac{1}{3} \left(3k - 2 \frac{S_0}{Y_0} \right) \gamma_k^k g^{\alpha\beta}. \tag{5.53}$$

The condition

$$S^{33} = 0, \tag{5.54}$$

which is Love's assumption, is applied through Eq. (3.107) and gives

$$\gamma_3^3 = \frac{2 \frac{S_0}{Y_0} - 3k}{4 \frac{S_0}{Y_0} + 3k} \gamma_\alpha^\alpha. \tag{5.55}$$

Next, substitution of Eq. (5.55) into Eq. (5.53) gives

$$\dot{S}^{\alpha\beta} = 2 \frac{\dot{S}_0}{\dot{\gamma}_0} \dot{\gamma}^{\alpha\beta} + 2 \frac{\dot{S}_0}{\dot{\gamma}_0} \left(\frac{3\dot{K} - 2 \frac{\dot{S}_0}{\dot{\gamma}_0}}{3\dot{K} + 4 \frac{\dot{S}_0}{\dot{\gamma}_0}} \right) \dot{\gamma}_\epsilon^\epsilon g^{\alpha\beta} \quad (5.56)$$

which reduces to the expressions for classical theory as given in reference [22].

Similarly, the stress-strain equation for the lower layer is

$$\ddot{S}^{\alpha\beta} = 2 \frac{\ddot{S}_0}{\ddot{\gamma}_0} \ddot{\gamma}^{\alpha\beta} + 2 \frac{\ddot{S}_0}{\ddot{\gamma}_0} \left(\frac{3\ddot{K} - 2 \frac{\ddot{S}_0}{\ddot{\gamma}_0}}{3\ddot{K} + 4 \frac{\ddot{S}_0}{\ddot{\gamma}_0}} \right) \ddot{\gamma}_\epsilon^\epsilon g^{\alpha\beta}. \quad (5.57)$$

Also, the inverse relations can be written immediately by referring to the equations of Sect. 3.3.4. From Eq. (3.96), and with

$$\dot{S}^{33} = \ddot{S}^{33} = 0,$$

the inverse relations are

$$\begin{aligned} \dot{\gamma}^{\alpha\beta} &= \frac{1}{2} \frac{\dot{\gamma}_0}{\dot{S}_0} \dot{S}^{\alpha\beta} + \frac{1}{3} \left(\frac{1}{3\dot{K}} - \frac{1}{2} \frac{\dot{\gamma}_0}{\dot{S}_0} \right) \dot{S}_\epsilon^\epsilon g^{\alpha\beta}, \\ \ddot{\gamma}^{\alpha\beta} &= \frac{1}{2} \frac{\ddot{\gamma}_0}{\ddot{S}_0} \ddot{S}^{\alpha\beta} + \frac{1}{3} \left(\frac{1}{3\ddot{K}} - \frac{1}{2} \frac{\ddot{\gamma}_0}{\ddot{S}_0} \right) \ddot{S}_\epsilon^\epsilon g^{\alpha\beta}, \\ \ddot{\gamma}^{\alpha 3} &= \frac{1}{2} \frac{\ddot{\gamma}_0}{\ddot{S}_0}, \end{aligned} \quad (5.58a)$$

$$\begin{aligned}\bar{\gamma}^{33} &= \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2} \frac{\bar{\gamma}_0}{\bar{S}_0} \right) \bar{S}_2 + \frac{1}{3} \left(\frac{1}{3k} + \frac{\bar{\gamma}_0}{\bar{S}_0} \right) \bar{S}_3, \\ {}''\gamma^{\alpha\beta} &= \frac{1}{2} \frac{{}''\gamma_0}{S_0} S^{\alpha\beta} + \frac{1}{3} \left(\frac{1}{3k} - \frac{1}{2} \frac{{}''\gamma_0}{S_0} \right) S^\epsilon_\epsilon g^{\alpha\beta}. \quad (5.58b)\end{aligned}$$

Naturally, the equations can be written in different forms; for example,

$$\begin{aligned}\dot{S}^{\alpha\beta} &= K^{\alpha\beta\gamma\delta} \dot{\gamma}_{\gamma\delta}, \\ \bar{S}^{\alpha\beta} &= \bar{E}^{\alpha\beta\gamma\delta} \bar{\gamma}_{\gamma\delta} + \bar{E}^{\alpha\beta 33} \bar{\gamma}_{33}, \\ \bar{S}^{\alpha 3} &= 2 \bar{E}^{\alpha 3 \gamma 3} \bar{\gamma}_{\gamma 3}, \\ \bar{S}^{33} &= \bar{E}^{\alpha\beta 33} \bar{\gamma}_{\alpha\beta} + \bar{E}^{3333} \bar{\gamma}_{33}, \\ {}''S^{\alpha\beta} &= {}''K^{\alpha\beta\gamma\delta} {}''\gamma_{\gamma\delta},\end{aligned} \quad (5.59)$$

where

$$K^{\alpha\beta\gamma\delta} = \frac{S_0}{\gamma_0} (g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}) + 2 \frac{S_0}{\gamma_0} \left(\frac{3k - 2 \frac{S_0}{\gamma_0}}{3k + 4 \frac{S_0}{\gamma_0}} \right) g^{\alpha\beta} g^{\gamma\delta}, \quad (5.60)$$

$$E^{ijkl} = \frac{S_0}{\gamma_0} (g^{ik} g^{jl} + g^{il} g^{jk}) + \frac{1}{3} \left(3k - 2 \frac{S_0}{\gamma_0} \right) g^{ij} g^{kl}. \quad (5.61)$$

5.2. Buckling Problem

The method of establishing the equations associated with buckling has been developed in Sect. 3.4. Therefore, it is easy to show that corresponding to Eqs. (5.4) the displacement functions Ψ_i associated with the buckling phenomenon for the upper and lower layers are, respectively,

$$\begin{aligned}\dot{\Psi}_i &= - \left(\bar{U}_{3,i} + \frac{\bar{h}}{2} \bar{\Psi}_{3,i} \right), \\ \ddot{\Psi}_i &= - \left(\bar{U}_{3,i} - \frac{\bar{h}}{2} \bar{\Psi}_{3,i} \right).\end{aligned}\tag{5.62}$$

Then, the infinitesimal displacements that occur during buckling for the three layers are

$$\begin{aligned}\delta \dot{V}_i &= \delta \bar{U}_i + \frac{\bar{h}}{2} \left(\delta \bar{\Psi}_i + \delta \bar{U}_{3,i} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,i} \right) \\ &\quad - \bar{\theta}^3 \left(\delta \bar{U}_{3,i} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,i} \right), \\ \delta \bar{V}_i &= \delta \bar{U}_i + \bar{\theta}^3 \delta \bar{\Psi}_i, \\ \delta \ddot{V}_i &= \delta \bar{U}_i - \frac{\bar{h}}{2} \left(\delta \bar{\Psi}_i + \delta \bar{U}_{3,i} - \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,i} \right) \\ &\quad - \bar{\theta}^3 \left(\delta \bar{U}_{3,i} - \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,i} \right).\end{aligned}\tag{5.63}$$

The variations of stress resultants have been established and are defined by Eqs. (3.114) - (3.119); however, the quantities δe^{i3} that appear in Eq. (3.118) have to be established for the current theory.

5.2.1. Field equations

The field equations are determined from equilibrium equations that result from Eqs. (5.26) - (5.29) by omitting the inertia terms. The first equilibrium equation for the stable position is

$$(\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta})|_{\beta} + \dot{p}^{\alpha} + \ddot{p}^{\alpha} + \ddot{p}^{\alpha} = 0. \quad (5.64)$$

By substituting the relations between the stable and unstable conditions, e.g.,

$$\dot{N}^{\alpha\beta} = \dot{N}^{\alpha\beta} + \delta \dot{m}^{\alpha\beta}$$

into Eq. (5.64), we obtain

$$\begin{aligned} & [(\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) + (\delta \dot{m}^{\alpha\beta} + \delta \ddot{m}^{\alpha\beta} + \delta \ddot{m}^{\alpha\beta})]|_{\beta} \\ & + (\dot{p}^{\alpha} + \ddot{p}^{\alpha} + \ddot{p}^{\alpha}) + (\delta \dot{p}^{\alpha} + \delta \ddot{p}^{\alpha} + \delta \ddot{p}^{\alpha}) = 0. \end{aligned} \quad (5.65)$$

Then, by subtracting the equilibrium equation for the unstable condition, i.e.,

$$(\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta})|_{\beta} + \dot{p}^{\alpha} + \ddot{p}^{\alpha} + \ddot{p}^{\alpha} = 0,$$

from Eq. (5.65), we obtain

$$(\dot{m}^{\alpha\beta} + \bar{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta})|_{\beta} + \dot{p}^{\alpha} + \bar{p}^{\alpha} + \ddot{p}^{\alpha} = 0 \quad (5.66)$$

after cancelling the δ . Eq. (5.66) is the first of four field equations; the remaining equations are determined by the identical procedure, and can be written by inspection from Eqs. (5.27) - (5.29)

$$\left[\frac{\hbar}{2} (\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right] |_{\beta} - \bar{g}^{\alpha} + \frac{\hbar}{2} (\dot{p}^{\alpha} - \ddot{p}^{\alpha}) + \bar{c}^{\alpha} = 0, \quad (5.67)$$

$$\begin{aligned} & \{ (\dot{N}^{\beta\gamma} + \bar{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) \bar{U}_{3,\gamma} + \left[\frac{\hbar}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{N}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} \\ & + (\dot{m}^{\beta\gamma} + \bar{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) \bar{u}_{3,\gamma} + \left[\frac{\hbar}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} \\ & - \frac{\hbar}{2} (\dot{p}^{\beta} - \ddot{p}^{\beta}) + \dot{c}^{\beta} + \bar{c}^{\beta} + \bar{q}^{\beta} \bar{\Psi}_3 + \bar{g}^{\beta} (1 + \bar{\Psi}_3) \} |_{\beta} \\ & - \left[\frac{\hbar}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) - \dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma} \right] |_{\beta\gamma} + \dot{p}^3 + \bar{p}^3 + \ddot{p}^3 = 0, \end{aligned} \quad (5.68)$$

$$\begin{aligned}
& \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\beta\gamma} - \ddot{N}^{\beta\gamma}) + \bar{M}^{\beta\gamma} \right] \bar{U}_{3,\gamma} + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\beta\gamma} + \ddot{N}^{\beta\gamma}) + \bar{K}^{\beta\gamma} \right] \bar{\Psi}_{3,\gamma} \right. \\
& + \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} - \ddot{m}^{\beta\gamma}) + \bar{m}^{\beta\gamma} \right] \bar{u}_{3,\gamma} + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) + \bar{k}^{\beta\gamma} \right] \bar{\psi}_{3,\gamma} \\
& - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{p}^\beta + \ddot{p}^\beta) - \dot{c}^\beta + \ddot{c}^\beta \right] \Big|_\beta - \bar{Q}^\alpha \bar{U}_{3,\alpha} - \bar{q}^\alpha \bar{u}_{3,\alpha} \\
& - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma}) - \dot{m}^{\beta\gamma} + \ddot{m}^{\beta\gamma} \right] \Big|_{\beta\gamma} + (\bar{T}^\beta_\beta - \bar{N}^{33}) \bar{\Psi}_3 \\
& + (\bar{t}^\beta_\beta - \bar{M}^{33}) (1 + \bar{\Psi}_3) + \frac{\bar{h}}{2} (\dot{p}^3 - \ddot{p}^3) + \bar{c}^3 = 0.
\end{aligned} \tag{5.69}$$

In Eqs. (5.68) and (5.69), the higher order terms such as

$$\delta \dot{m}^{\alpha\beta} \delta \bar{U}_{3,\alpha}$$

have been neglected.

To complete this section, the quantities P^i and C^i must be established. First, consider the term ${}_2\dot{p}^\alpha$ which is defined by Eq. (5.23); for the stable position, the corresponding quantity is

$${}_2\dot{p}^\alpha = \dot{S}^{\alpha 3} = \dot{S}^{\alpha 3} + \delta \dot{x}^{\alpha 3}, \quad \text{AT } \dot{\theta}^3 = \frac{\dot{h}}{2}.$$

Since

$${}_2\dot{p}^\alpha = {}_2\dot{p}^\alpha + \delta {}_2\dot{p}^\alpha,$$

and

$${}_2'p^\alpha = {}_2's^{\alpha 3}, \quad {}_{AT}'\theta^3 = \frac{{}_2'h}{2},$$

we obtain

$$\delta {}_2'p^\alpha = \delta {}_2's^{\alpha 3}, \quad {}_{AT}'\theta^3 = \frac{{}_2'h}{2}.$$

In a similar manner, we obtain from Eq. (5.24)

$$\delta {}_2'p^3 = \delta {}_2'e^{33}, \quad {}_{AT}'\theta^3 = \frac{{}_2'h}{2},$$

where

$$\begin{aligned} \delta {}_2'e^{33} = & {}_2's^{\alpha 3} \left(\delta \bar{U}_{3,\alpha} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha} \right) + \delta {}_2's^{33} \\ & + \delta {}_2's^{\alpha 3} \left(\bar{U}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\Psi}_{3,\alpha} \right). \end{aligned}$$

Summarizing, we define the following:

$$\begin{aligned} \delta \left\{ \begin{matrix} {}_m'p^\alpha, {}_m'\bar{c}^\alpha \\ {}_m'\bar{p}^\alpha, {}_m'\bar{c}^\alpha \\ {}_m''p^\alpha, {}_m''c^\alpha \end{matrix} \right\} = \delta \left\{ \begin{matrix} {}_2's^{\alpha 3}, \bar{\theta}^3 {}_2's^{\alpha 3} \\ \bar{s}^{\alpha 3}, \bar{\theta}^3 \bar{s}^{\alpha 3} \\ {}_2''s^{\alpha 3}, \bar{\theta}^3 {}_2''s^{\alpha 3} \end{matrix} \right\} \Bigg\} {}_{AT} \left\{ \begin{matrix} {}_2'\theta^3 \\ \bar{\theta}^3 \\ {}_2''\theta^3 \end{matrix} \right\} = (-1)^m \left\{ \begin{matrix} \frac{{}_2'h}{2} \\ \bar{h} \\ {}_2''h \end{matrix} \right\} \\ \delta \left\{ \begin{matrix} {}_m'p^3, {}_m'\bar{c}^3 \\ {}_m'\bar{p}^3, {}_m'\bar{c}^3 \\ {}_m''p^3, {}_m''c^3 \end{matrix} \right\} = \delta \left\{ \begin{matrix} {}_2'e^{33}, \bar{\theta}^3 {}_2'e^{33} \\ \bar{e}^{33}, \bar{\theta}^3 \bar{e}^{33} \\ {}_2''e^{33}, \bar{\theta}^3 {}_2''e^{33} \end{matrix} \right\} \Bigg\} {}_{AT} \left\{ \begin{matrix} {}_2'\theta^3 \\ \bar{\theta}^3 \\ {}_2''\theta^3 \end{matrix} \right\} = (-1)^m \left\{ \begin{matrix} \frac{{}_2'h}{2} \\ \bar{h} \\ {}_2''h \end{matrix} \right\} \end{aligned} \quad (5.70)$$

which are used in conjunction with Eqs. (3.119). In the above,

$$\begin{aligned}
 \delta' e^{33} &= \delta' \bar{u}^{\alpha 3} \left(\bar{u}_{3,\alpha} + \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + \delta' \bar{u}^{33} \\
 &\quad + \delta' \bar{s}^{\alpha 3} \left(\delta \bar{U}_{3,\alpha} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha} \right), \\
 \delta \bar{e}^{33} &= \delta \bar{u}^{\alpha 3} \left(\bar{u}_{3,\alpha} + \bar{\theta}^3 \bar{\psi}_{3,\alpha} \right) + \delta \bar{u}^{33} (1 + \bar{\psi}_3) \\
 &\quad + \delta \bar{s}^{\alpha 3} \left(\delta \bar{U}_{3,\alpha} + \bar{\theta}^3 \delta \bar{\Psi}_{3,\alpha} \right) + \delta \bar{s}^{33} \delta \bar{\Psi}_3, \\
 \delta'' e^{33} &= \delta'' \bar{u}^{\alpha 3} \left(\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\psi}_{3,\alpha} \right) + \delta'' \bar{u}^{33} \\
 &\quad + \delta'' \bar{s}^{\alpha 3} \left(\delta \bar{U}_{3,\alpha} - \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha} \right).
 \end{aligned} \tag{5.71}$$

5.2.2. Boundary conditions

For buckling, the boundary conditions are obtained from the boundary conditions for the two equilibrium positions. First, consider the stress boundary conditions. For the top surface, the equations for the stable and unstable positions (refer to Sect. 5.1.2) are, respectively,

$${}_2' \dot{p}_*^i - {}_2' \dot{p}^{\bar{i}} = 0, \quad {}_2' \dot{c}_*^i - {}_2' \dot{c}^{\bar{i}} = 0, \tag{5.72a}$$

and

$${}_2' p_*^i - {}_2' p^i = 0, \quad {}_2' c_*^i - {}_2' c^i = 0, \tag{5.72b}$$

where by definition, from Eqs. (3.49) and (3.110),

$$\left. \begin{aligned} {}_2'\dot{p}_*^i &= {}_2'\dot{\tilde{s}}_*^i = {}_2'\tilde{s}_*^i + \delta {}_2'\tilde{s}_*^i \\ {}_2'\dot{p}^i &= {}_2'\dot{E}^{33} = {}_2'E^{33} + \delta {}_2'e^{33} \end{aligned} \right\} \text{ AT } \bar{\theta}^3 = \frac{\bar{h}}{2} + h,$$

or

$$\begin{aligned} {}_2'\dot{p}_*^i &= {}_2'p_*^i + \delta {}_2'p_*^i, \\ {}_2'\dot{p}^i &= {}_2'p^i + \delta {}_2'p^i. \end{aligned}$$

Similarly,

$$\begin{aligned} {}_2'\dot{c}_*^i &= {}_2'c_*^i + \delta {}_2'c_*^i, \\ {}_2'\dot{c}^i &= {}_2'c^i + \delta {}_2'c^i. \end{aligned}$$

Therefore, subtraction of Eqs. (5.72b) from Eqs. (5.72a) yields

$$\delta {}_2'p_*^i - \delta {}_2'p^i = 0, \quad \delta {}_2'c_*^i - \delta {}_2'c^i = 0,$$

which are the stress boundary conditions for the top surface during buckling.

For the edge surface, the stress boundary conditions are obtained from Eqs. (5.33) - (5.36), (5.39) and (5.40). For example, in the stable position, Eq. (5.33) goes over to

$${}_0\dot{S}^\alpha + {}_0\ddot{S}^\alpha + {}_0'''S^\alpha = {}_0\bar{m}_\beta ({}_0\dot{N}^{\alpha\beta} + {}_0\ddot{N}^{\alpha\beta} + {}_0'''N^{\alpha\beta}) \quad (5.73)$$

where

$${}_0\dot{S}^\alpha = {}_0\dot{S}^\alpha + \delta {}_0\dot{S}^\alpha,$$

$${}_0\dot{N}^{\alpha\beta} = {}_0\dot{N}^{\alpha\beta} + \delta {}_0\dot{N}^{\alpha\beta}, \text{ etc.}$$

Subtracting the equation corresponding to Eq. (5.73) for the unstable position, i.e., Eq. (5.33), from Eq. (5.73) gives

$$\delta {}_0\dot{S}^\alpha + \delta {}_0\ddot{S}^\alpha + \delta {}_0'''S^\alpha = {}_0\bar{m}_\beta (\delta {}_0\dot{N}^{\alpha\beta} + \delta {}_0\ddot{N}^{\alpha\beta} + \delta {}_0'''N^{\alpha\beta})$$

as a boundary condition. The remaining stress boundary conditions are calculated as above; in summary, the stress boundary conditions are as follows (for convenience, the δ has been cancelled):

on the top surface,

$${}_2P_*^i - {}_2P^i = 0, \quad {}_2C_*^i - {}_2C^i = 0; \quad (5.74)$$

on the bottom surface,

$${}_1P_*^i + {}_1P^i = 0, \quad {}_1C_*^i + {}_1C^i = 0; \quad (5.75)$$

on the edge surface,

$${}'_0\dot{\bar{e}}^\alpha + {}'_0\bar{\dot{e}}^\alpha + {}''_0\dot{\bar{e}}^\alpha = {}_0\bar{m}_\beta ({}'_0m^{\alpha\beta} + \bar{m}^{\alpha\beta} + {}''_0m^{\alpha\beta}), \quad (5.76)$$

$$\frac{\bar{h}}{2} ({}'_0\dot{\bar{e}}^\alpha - {}''_0\dot{\bar{e}}^\alpha) + {}_0\bar{\dot{e}}^\alpha = {}_0\bar{m}_\beta \left[\frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right], \quad (5.77)$$

$${}'_0\dot{\bar{e}}^\alpha + {}''_0\dot{\bar{e}}^\alpha - \frac{\bar{h}}{2} ({}'_0\dot{\bar{e}}^\alpha - {}''_0\dot{\bar{e}}^\alpha) = {}_0\bar{m}_\beta \left[{}'_0m^{\alpha\beta} + {}''_0m^{\alpha\beta} - \frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) \right], \quad (5.78)$$

$${}'_0\dot{\bar{e}}^\alpha - {}''_0\dot{\bar{e}}^\alpha - \frac{\bar{h}}{2} ({}'_0\dot{\bar{e}}^\alpha + {}''_0\dot{\bar{e}}^\alpha) = {}_0\bar{m}_\beta \left[{}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta} - \frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} + {}''_0m^{\alpha\beta}) \right], \quad (5.79)$$

$$\begin{aligned} {}'_0\dot{\bar{e}}^3 + {}'_0\bar{\dot{e}}^3 + {}''_0\dot{\bar{e}}^3 = {}_0\bar{m}_\beta \bigg\{ & ({}'_0N^{\alpha\beta} + \bar{N}^{\alpha\beta} + {}''_0N^{\alpha\beta}) \bar{U}_{3,\alpha} \\ & + \left[\frac{\bar{h}}{2} ({}'_0N^{\alpha\beta} - {}''_0N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} + ({}'_0m^{\alpha\beta} + \bar{m}^{\alpha\beta} + {}''_0m^{\alpha\beta}) \bar{U}_{3,\alpha} \\ & + \left[\frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} - \left[\frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) - {}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta} \right] \Big|_\alpha \\ & - \frac{\bar{h}}{2} ({}'_0p^\beta - {}''_0p^\beta) + \dot{C}^\beta + {}''C^\beta + \bar{Q}^\beta \bar{\Psi}_3 + \bar{q}^\beta (1 + \bar{\Psi}_3) \bigg\}, \end{aligned} \quad (5.80)$$

$$\begin{aligned} \frac{\bar{h}}{2} ({}'_0\dot{\bar{e}}^3 - {}''_0\dot{\bar{e}}^3) + {}_0\bar{\dot{e}}^3 = {}_0\bar{m}_\beta \bigg\{ & \left[\frac{\bar{h}}{2} ({}'_0N^{\alpha\beta} - {}''_0N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{U}_{3,\alpha} \\ & + \left[\left(\frac{\bar{h}}{2} \right)^2 ({}'_0N^{\alpha\beta} + {}''_0N^{\alpha\beta}) + \bar{K}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} + \left[\frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right] \bar{U}_{3,\alpha} \\ & + \left[\left(\frac{\bar{h}}{2} \right)^2 ({}'_0m^{\alpha\beta} + {}''_0m^{\alpha\beta}) + \bar{k}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} ({}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta}) - {}'_0m^{\alpha\beta} - {}''_0m^{\alpha\beta} \right] \Big|_\alpha \\ & - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} ({}'_0p^\beta + {}''_0p^\beta) - \dot{C}^\beta + {}''C^\beta \right] + \bar{T}^\beta \bar{\Psi}_3 + \bar{t}^\beta (1 + \bar{\Psi}_3) \bigg\}. \end{aligned} \quad (5.81)$$

In the above equations, the definitions of Eqs. (3.28) - (3.30), (3.114) - (3.117), (3.119), (3.144), (3.145) and (5.70) apply to the various quantities (i.e., $'N^{\alpha\beta}$, $'n^{\alpha\beta}$, etc.); also, all higher order terms have been neglected.

The displacement boundary conditions are obtained in the exact same way as for the general theory (see Sect. 3.4.2), and the results are as follows:

on the top surface,

$$'V_i - '\tilde{V}_i = 0; \quad (5.82a)$$

on the bottom surface,

$$''V_i - ''\tilde{V}_i = 0; \quad (5.82b)$$

and on the edge surface,

$$\begin{aligned} 'V_i - '\tilde{V}_i &= 0, \\ \bar{V}_i - \bar{\tilde{V}}_i &= 0, \\ ''V_i - ''\tilde{V}_i &= 0. \end{aligned} \quad (5.82c)$$

Again, the infinitesimal δ quantity has been cancelled for convenience.

5.2.3. Variation of strain

The strain variation that occurs during buckling is calculated by the method of Sect. 3.4.3. From Eqs. (5.42), the strain tensors for the stable position are

$$\begin{aligned}
 \dot{\gamma}_{\alpha\beta} &= \frac{1}{2} \left(\dot{v}_{\alpha/\beta} + \dot{v}_{\beta/\alpha} + \dot{v}_{3,\alpha} \dot{v}_{3,\beta} \right), \\
 \bar{\dot{\gamma}}_{\alpha\beta} &= \frac{1}{2} \left(\bar{\dot{v}}_{\alpha/\beta} + \bar{\dot{v}}_{\beta/\alpha} + \bar{\dot{v}}_{3,\alpha} \bar{\dot{v}}_{3,\beta} \right), \\
 \dot{\gamma}_{\alpha 3} &= \frac{1}{2} \left(\dot{v}_{\alpha,3} + \dot{v}_{3,\alpha} + \dot{v}_{3,\alpha} \dot{v}_{3,3} \right), \\
 \dot{\gamma}_{33} &= \dot{v}_{3,3} + \frac{1}{2} \dot{v}_{3,3} \dot{v}_{3,3}, \\
 \bar{\dot{\gamma}}_{\alpha\beta} &= \frac{1}{2} \left(\bar{\dot{v}}_{\alpha/\beta} + \bar{\dot{v}}_{\beta/\alpha} + \bar{\dot{v}}_{3,\alpha} \bar{\dot{v}}_{3,\beta} \right),
 \end{aligned} \tag{5.83}$$

for the three layers. By substituting the equations that relate the displacements in the two equilibrium positions, e.g.,

$$\dot{v}_i = v_i + \delta v_i,$$

the relations between the strain tensors for the two positions are obtained as follows:

$$\begin{aligned}
 \dot{\gamma}_{\alpha\beta} &= \bar{\gamma}_{\alpha\beta} + \delta \bar{\gamma}_{\alpha\beta} + \text{higher order terms;} \\
 \bar{\dot{\gamma}}_{\alpha\beta} &= \bar{\gamma}_{\alpha\beta} + \delta \bar{\gamma}_{\alpha\beta} + \text{higher order terms;}
 \end{aligned}$$

$$\dot{\bar{\gamma}}_{\alpha 3} = \bar{\gamma}_{\alpha 3} + \delta \bar{\Gamma}_{\alpha 3} + \text{higher order terms};$$

$$\dot{\bar{\gamma}}_{33} = \bar{\gamma}_{33} + \delta \bar{\Gamma}_{33} + \text{higher order terms};$$

$${}''\dot{\bar{\gamma}}_{\alpha\beta} = {}''\bar{\gamma}_{\alpha\beta} + \delta {}''\bar{\Gamma}_{\alpha\beta} + \text{higher order terms};$$

where the strain variations are

$$\delta \bar{\Gamma}'_{\alpha\beta} = \frac{1}{2} (\delta \bar{V}'_{\alpha|\beta} + \delta \bar{V}'_{\beta|\alpha} + \bar{v}'_{3,\alpha} \delta \bar{V}'_{3,\beta} + \delta \bar{V}'_{3,\alpha} \bar{v}'_{3,\beta}),$$

$$\delta \bar{\Gamma}_{\alpha\beta} = \frac{1}{2} (\delta \bar{V}_{\alpha|\beta} + \delta \bar{V}_{\beta|\alpha} + \bar{v}_{3,\alpha} \delta \bar{V}_{3,\beta} + \delta \bar{V}_{3,\alpha} \bar{v}_{3,\beta}),$$

$$\delta \bar{\Gamma}_{\alpha 3} = \frac{1}{2} (\delta \bar{V}_{\alpha,3} + \delta \bar{V}_{3,\alpha} + \bar{v}_{3,\alpha} \delta \bar{V}_{3,3} + \delta \bar{V}_{3,\alpha} \bar{v}_{3,3}),$$

$$\delta \bar{\Gamma}_{33} = \delta \bar{V}_{3,3} + \bar{v}_{3,3} \delta \bar{V}_{3,3},$$

$$\delta {}''\bar{\Gamma}_{\alpha\beta} = \frac{1}{2} (\delta {}''\bar{V}_{\alpha|\beta} + \delta {}''\bar{V}_{\beta|\alpha} + {}''\bar{v}_{3,\alpha} \delta {}''\bar{V}_{3,\beta} + \delta {}''\bar{V}_{3,\alpha} {}''\bar{v}_{3,\beta}).$$

By substituting the displacement functions, Eqs. (5.7) and (5.63), the above become

$$\delta \bar{\Gamma}'_{\alpha\beta} = \delta {}_0\bar{\Gamma}'_{\alpha\beta} + \bar{\Theta}^3 \delta {}_1\bar{\Gamma}'_{\alpha\beta},$$

$$\delta \bar{\Gamma}_{\alpha\beta} = \delta {}_0\bar{\Gamma}_{\alpha\beta} + \bar{\Theta}^3 \delta {}_1\bar{\Gamma}_{\alpha\beta} + (\bar{\Theta}^3)^2 \delta {}_2\bar{\Gamma}_{\alpha\beta}, \quad (5.84a)$$

$$\delta \bar{\Gamma}_{\alpha 3} = \delta {}_0\bar{\Gamma}_{\alpha 3} + \bar{\Theta}^3 \delta {}_1\bar{\Gamma}_{\alpha 3},$$

$$\delta \bar{\Gamma}_{33} = \delta_o \bar{\Gamma}_{33},$$

(5.84b)

$$\delta'' \bar{\Gamma}_{\alpha\beta} = \delta_o'' \bar{\Gamma}_{\alpha\beta} + \bar{\theta}^3 \delta_o' \bar{\Gamma}_{\alpha\beta},$$

where

$$\begin{aligned} {}_o' \bar{\Gamma}_{\alpha\beta} = & \frac{1}{2} \left[\bar{U}_{\alpha|\beta} + \bar{U}_{\beta|\alpha} + \frac{\hbar}{2} (\bar{\Psi}_{\alpha|\beta} + \bar{\Psi}_{\beta|\alpha} + 2 \bar{U}_{3,\alpha\beta} \right. \\ & + \bar{h} \bar{\Psi}_{3,\alpha\beta}) + (\bar{u}_{3,\alpha} + \frac{\hbar}{2} \bar{\Psi}_{3,\alpha})(\bar{U}_{3,\beta} + \frac{\hbar}{2} \bar{\Psi}_{3,\beta}) \\ & \left. + (\bar{U}_{3,\alpha} + \frac{\hbar}{2} \bar{\Psi}_{3,\alpha})(\bar{u}_{3,\beta} + \frac{\hbar}{2} \bar{\Psi}_{3,\beta}) \right], \end{aligned} \quad (5.85)$$

$${}_i' \bar{\Gamma}_{\alpha\beta} = - (\bar{U}_{3,\alpha\beta} + \frac{\hbar}{2} \bar{\Psi}_{3,\alpha\beta}), \quad (5.86)$$

$${}_o \bar{\Gamma}_{\alpha\beta} = \frac{1}{2} (\bar{U}_{\alpha|\beta} + \bar{U}_{\beta|\alpha} + \bar{u}_{3,\alpha} \bar{U}_{3,\beta} + \bar{U}_{3,\alpha} \bar{u}_{3,\beta}), \quad (5.87)$$

$$\begin{aligned} {}_i \bar{\Gamma}_{\alpha\beta} = & \frac{1}{2} (\bar{\Psi}_{\alpha|\beta} + \bar{\Psi}_{\beta|\alpha} + \bar{\Psi}_{3,\alpha} \bar{U}_{3,\beta} + \bar{U}_{3,\alpha} \bar{\Psi}_{3,\beta} \\ & + \bar{u}_{3,\alpha} \bar{\Psi}_{3,\beta} + \bar{\Psi}_{3,\alpha} \bar{u}_{3,\beta}), \end{aligned} \quad (5.88)$$

$${}_2 \bar{\Gamma}_{\alpha\beta} = \frac{1}{2} (\bar{\Psi}_{3,\alpha} \bar{\Psi}_{3,\beta} + \bar{\Psi}_{3,\alpha} \bar{\Psi}_{3,\beta}), \quad (5.89)$$

$${}_o \bar{\Gamma}_{\alpha 3} = \frac{1}{2} (\bar{\Psi}_{\alpha} + \bar{U}_{3,\alpha} + \bar{u}_{3,\alpha} \bar{\Psi}_3 + \bar{U}_{3,\alpha} \bar{\Psi}_3), \quad (5.90)$$

$${}_i \bar{\Gamma}_{\alpha 3} = \frac{1}{2} (\bar{\Psi}_{3,\alpha} + \bar{\Psi}_3 \bar{\Psi}_{3,\alpha} + \bar{\Psi}_3 \bar{\Psi}_{3,\alpha}), \quad (5.91)$$

$${}_0\bar{\Gamma}_{33} = \bar{\Psi}_3 + \bar{\Psi}_3 \bar{\Psi}_3, \quad (5.92)$$

$$\begin{aligned} {}_0''\Gamma_{\alpha\beta} = & \frac{1}{2} \left[\bar{U}_{\alpha|\beta} + \bar{U}_{\beta|\alpha} - \frac{\bar{h}}{2} (\bar{\Psi}_{\alpha|\beta} + \bar{\Psi}_{\beta|\alpha} + 2 \bar{U}_{3,\alpha\beta} \right. \\ & - \bar{h} \bar{\Psi}_{3,\alpha\beta}) + (\bar{u}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\alpha}) (\bar{U}_{3,\beta} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\beta}) \\ & \left. + (\bar{U}_{3,\alpha} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\alpha}) (\bar{u}_{3,\beta} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\beta}) \right], \quad (5.93) \end{aligned}$$

$${}_1'\Gamma_{\alpha\beta} = - (\bar{U}_{3,\alpha\beta} - \frac{\bar{h}}{2} \bar{\Psi}_{3,\alpha\beta}). \quad (5.94)$$

Note that the strain variations

$$\delta'\Gamma_{\alpha 3}, \quad \delta'\Gamma_{33}, \quad \delta''\Gamma_{\alpha 3}, \quad \delta''\Gamma_{33}$$

are all zero.

5.2.4. Variation of stress

Consider the middle layer first. In Sect. 5.1.4, all the stress components were found to be finite; furthermore, the stress-strain equations are identically the same as for the general theory, given in Sect. 3.3.4. Therefore, the variation of stress for the middle layer is also identical with the general theory as developed in Sec. 3.4.4. Thus, from Eqs. (3.162) and (3.163),

$$\delta \bar{\sigma}^{ij} = \bar{M}^{ijkl} \delta \bar{\Gamma}_{kl} \quad (5.95)$$

where

$$\bar{M}^{ijkl} = \bar{E}^{ijkl} + \frac{2}{3\bar{S}_0^2} \left(\frac{\delta \bar{S}_0}{\delta \bar{\gamma}_0} - \frac{\bar{S}_0}{\bar{\gamma}_0} \right) (\bar{S}^{ij} - \bar{S}g^{ij})(\bar{S}^{kl} - \bar{S}g^{kl}); \quad (5.96)$$

or in subtensor form, Eqs. (3.165) are applicable.

Next, consider the upper layer; two methods may be used to establish the stress variation. One approach is to use the stress-strain equation, Eq. (5.59), of Sect. 5.1.4, i.e.,

$$\bar{S}^{\alpha\beta} = \bar{K}^{\alpha\beta\gamma\delta} \bar{\gamma}_{\gamma\delta},$$

and operate upon it. The second method is to use the general equations of Sect. 3.4.4 and reduce them for this special case. The second method is more general and, therefore, will be employed; the equations that will be used are Eqs. (3.165).

Recall that the stress-strain equation for the upper layer was developed from the condition that $\bar{S}^{33} = 0$, which implies that

$$\delta \bar{S}^{33} = 0. \quad (5.97)$$

Substitution of Eq. (5.97) into Eq. (3.165) gives (the primes are omitted for convenience)

$$M^{\gamma\delta 33} \delta \gamma_{\gamma\delta} + 2M^{\gamma 333} \delta \gamma_{\gamma 3} + M^{3333} \delta \gamma_{33} = 0.$$

From Eqs. (3.166) and (3.108), it is seen that M^{3333} is a scalar quantity, namely,

$$k + 4 \frac{S_0}{Y_0} + \frac{2}{3S_0^2} \left(\frac{\delta S_0}{\delta Y_0} - \frac{S_0}{Y_0} \right) \left(\frac{1}{3} S_0^\alpha \right)^2$$

where $S^{33} = 0$ has been used. Therefore,

$$\delta \Gamma_{33} = \frac{M^{\gamma\delta 33}}{M^{3333}} \delta \Gamma_{\gamma\delta} - \frac{M^{\gamma 333}}{M^{3333}} \delta \Gamma_{\gamma 3}. \quad (5.98)$$

Substitution of the above into Eqs. (3.165) gives

$$\begin{aligned} \delta \mathcal{L}^{\alpha\beta} &= \left(M^{\alpha\beta\gamma\delta} - \frac{M^{\alpha\beta 33} M^{\gamma\delta 33}}{M^{3333}} \right) \delta \Gamma_{\gamma\delta} \\ &\quad + \left(2M^{\alpha\beta\gamma 3} - \frac{M^{\alpha\beta 33} M^{\gamma 333}}{M^{3333}} \right) \delta \Gamma_{\gamma 3}, \\ \delta \mathcal{L}^{\alpha 3} &= \left(M^{\alpha 3\gamma\delta} - \frac{M^{\alpha 333} M^{\gamma\delta 33}}{M^{3333}} \right) \delta \Gamma_{\gamma\delta} \\ &\quad + \left(2M^{\alpha 3\gamma 3} - \frac{M^{\alpha 333} M^{\gamma 333}}{M^{3333}} \right) \delta \Gamma_{\gamma 3}. \end{aligned} \quad (5.99)$$

After some lengthy but straightforward calculations the above can be written as

$$\begin{aligned} \delta \mathcal{L}^{\alpha\beta} &= R^{\alpha\beta\gamma\delta} \delta \Gamma_{\gamma\delta} + 2R^{\alpha\beta\gamma 3} \delta \Gamma_{\gamma 3}, \\ \delta \mathcal{L}^{\alpha 3} &= R^{\alpha 3\gamma\delta} \delta \Gamma_{\gamma\delta} + 2R^{\alpha 3\gamma 3} \delta \Gamma_{\gamma 3}, \end{aligned} \quad (5.100)$$

where

$$R^{\alpha i \gamma j} = A^{\alpha i \gamma j} + B^{\alpha i \gamma j}, \quad (5.101)$$

$$A^{\alpha i \gamma j} = \frac{S_0}{\gamma_0} (g^{\alpha \gamma} g^{ij} + g^{\alpha j} g^{i \gamma}) + 2 \frac{S_0}{\gamma_0} \left(\frac{3k-2}{3k+4} \frac{S_0}{\gamma_0} \right) g^{\alpha i} g^{\gamma j}, \quad (5.102)$$

$$B^{\alpha i \gamma j} = \phi \left[\left(3k+4 \frac{S_0}{\gamma_0} \right) S^{\alpha i} - 2 \frac{S_0}{\gamma_0} S^{\delta}_{\delta} g^{\alpha i} \right] \times \\ \times \left[\left(3k+4 \frac{S_0}{\gamma_0} \right) S^{\gamma j} - 2 \frac{S_0}{\gamma_0} S^{\epsilon}_{\epsilon} g^{\gamma j} \right], \quad (5.103)$$

$$\phi = \frac{\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0}}{\frac{3}{2} S_0^2 \left(3k+4 \frac{S_0}{\gamma_0} \right) \left[3k+4 \frac{S_0}{\gamma_0} + \frac{2}{9 S_0^2} \left(\frac{\delta S_0}{\delta \gamma_0} - \frac{S_0}{\gamma_0} \right) S^{\alpha}_{\alpha} S^{\beta}_{\beta} \right]}. \quad (5.104)$$

The above are the general stress variation equations under the conditions

$$S^{33} = \delta \mathcal{A}^{33} = 0.$$

Note the symmetry relations

$$R^{\alpha i \gamma j} = R^{i \alpha \gamma j} = R^{\alpha i j \gamma} = R^{\gamma j \alpha i}. \quad (5.105)$$

For the more restrictive theory under consideration, the conditions

$$\dot{S}^{\alpha 3} = \dot{Y}_{\alpha 3} = 0,$$

$$\delta \dot{S}^{\alpha 3} = \delta \dot{\bar{Y}}_{\alpha 3} = 0,$$

must be applied to the above equations. Thus, for the upper layer, the stress variations are

$$\delta \dot{S}^{\alpha\beta} = \dot{R}^{\alpha\beta\gamma\delta} \delta \dot{\bar{\Gamma}}_{\gamma\delta}. \quad (5.106)$$

Similarly, for the lower layer,

$$\delta \dot{S}^{\alpha\beta} = \dot{R}^{\alpha\beta\gamma\delta} \delta \dot{\bar{\Gamma}}_{\gamma\delta}. \quad (5.107)$$

When reduced to classical theory, the above agrees with the equations given by Teodosiadis, Langhaar, and Smith [22].

5.3. Modified Buckling Problem

As discussed in Sect. 3.5, the modified buckling problem is formulated by neglecting the effects of the displacements prior to buckling. Therefore, the equations of Sect. 5.2 are applicable when the displacement functions $\bar{u}_3$ and $\bar{\psi}_3$ are omitted; note that this is the same as assuming that the plate is flat prior to buckling.

5.3.1. Field equations

In view of the above, the field equations as obtained from Sect. 5.2.1 are as follows:

$$(\dot{m}^{\alpha\beta} + \bar{m}^{\alpha\beta} + {}''m^{\alpha\beta})|_{\beta} + \dot{p}^{\alpha} + \bar{p}^{\alpha} + {}''p^{\alpha} = 0; \quad (5.108)$$

$$\left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - {}''m^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right] |_{\beta} - \bar{g}^{\alpha} + \frac{\bar{h}}{2} (\dot{p}^{\alpha} - {}''p^{\alpha}) + \bar{c}^{\alpha} = 0; \quad (5.109)$$

$$\begin{aligned} & \{ (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + {}''N^{\alpha\beta}) \bar{U}_{3,\alpha} - \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - {}''N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} \\ & - \frac{\bar{h}}{2} (\dot{p}^{\beta} - {}''p^{\beta}) + \dot{c}^{\beta} + {}''c^{\beta} + \bar{Q}^{\beta} \bar{\Psi}_3 + \bar{g}^{\beta} \} |_{\beta} \end{aligned} \quad (5.110)$$

$$- \left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - {}''m^{\alpha\beta}) - \dot{m}^{\alpha\beta} - {}''m^{\alpha\beta} \right] |_{\alpha\beta} + \dot{p}^3 + \bar{p}^3 + {}''p^3 = 0;$$

$$\begin{aligned} & \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - {}''N^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{U}_{3,\alpha} + \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\alpha\beta} + {}''N^{\alpha\beta}) + \bar{K}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} \right. \\ & - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{p}^{\beta} + {}''p^{\beta}) - \dot{c}^{\beta} + {}''c^{\beta} \right] \} |_{\beta} - \bar{Q}^{\alpha} \bar{U}_{3,\alpha} + \bar{t}^{\beta} |_{\beta} - \bar{m}^{33} \\ & - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} + {}''m^{\alpha\beta}) - \dot{m}^{\alpha\beta} + {}''m^{\alpha\beta} \right] |_{\alpha\beta} + (\bar{T}^{\beta} |_{\beta} - \bar{N}^{33}) \bar{\Psi}_3 \\ & + \frac{\bar{h}}{2} (\dot{p}^3 - {}''p^3) + \bar{c}^3 = 0. \end{aligned} \quad (5.111)$$

In addition, the δe^{33} quantities, which are employed in the definitions of $\delta_n p^3$, are obtained from Eqs. (5.71) and become

$$\delta' e^{33} = \delta' \bar{e}^{33} + \dot{S}^{\alpha 3} \left(\delta \bar{U}_{3,\alpha} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha} \right), \quad (5.112a)$$

$$\delta \bar{e}^{33} = \delta \bar{e}^{33} + \bar{S}^{\alpha 3} (\delta \bar{U}_{3,\alpha} + \bar{\theta}^3 \delta \bar{\Psi}_{3,\alpha}) + \bar{S}^{33} \bar{\Psi}_3, \quad (5.112b)$$

$$\delta e''^{33} = \delta e''^{33} + \bar{S}^{\alpha 3} (\delta \bar{U}_{3,\alpha} - \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha}).$$

5.3.2. Boundary conditions

By neglecting the $\bar{u}_3$ and $\bar{\psi}_3$ terms from the equations of Sect. 5.2.2, the stress boundary conditions on the edge surface become

$${}_o\dot{e}^\alpha + {}_o\bar{e}^\alpha + {}_o\ddot{e}^\alpha = {}_o\bar{m}_\beta (\dot{m}^{\alpha\beta} + \bar{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta}), \quad (5.113)$$

$$\frac{\bar{h}}{2} ({}_o\dot{e}^\alpha - {}_o\ddot{e}^\alpha) + {}_o\bar{e}^\alpha = {}_o\bar{m}_\beta \left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta}) + \bar{m}^{\alpha\beta} \right], \quad (5.114)$$

$${}_o\dot{e}^\alpha + {}_o\ddot{e}^\alpha - \frac{\bar{h}}{2} ({}_o\dot{e}^\alpha - {}_o\ddot{e}^\alpha) = {}_o\bar{m}_\beta \left[\dot{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta} - \frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta}) \right], \quad (5.115)$$

$${}_o\dot{e}^\alpha - {}_o\ddot{e}^\alpha - \frac{\bar{h}}{2} ({}_o\dot{e}^\alpha + {}_o\ddot{e}^\alpha) = {}_o\bar{m}_\beta \left[\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta} - \frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta}) \right], \quad (5.116)$$

$$\begin{aligned} {}_o\dot{e}^3 + {}_o\bar{e}^3 + {}_o\ddot{e}^3 = {}_o\bar{m}_\beta \left\{ \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} \right. \\ \left. + (\dot{N}^{\alpha\beta} + \bar{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) \bar{U}_{3,\alpha} - \left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta}) - \dot{m}^{\alpha\beta} - \ddot{m}^{\alpha\beta} \right] \right|_\alpha \\ \left. + \bar{Q}^\beta \bar{\Psi}_3 + \bar{q}^\beta - \frac{\bar{h}}{2} (\dot{p}^\beta - \ddot{p}^\beta) + \dot{c}^\beta + \ddot{c}^\beta \right\}, \quad (5.117) \end{aligned}$$

$$\begin{aligned} \frac{\bar{h}}{2} ({}_o\dot{e}^3 - {}_o\ddot{e}^3) + {}_o\bar{e}^3 = {}_o\bar{m}_\beta \left\{ \left[\left(\frac{\bar{h}}{2} \right)^2 (\dot{N}^{\alpha\beta} + \ddot{N}^{\alpha\beta}) + \bar{K}^{\alpha\beta} \right] \bar{\Psi}_{3,\alpha} \right. \\ \left. + \left[\frac{\bar{h}}{2} (\dot{N}^{\alpha\beta} - \ddot{N}^{\alpha\beta}) + \bar{M}^{\alpha\beta} \right] \bar{U}_{3,\alpha} - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta}) - \dot{m}^{\alpha\beta} + \ddot{m}^{\alpha\beta} \right] \right|_\alpha \\ \left. + \bar{T}^\beta \bar{\Psi}_3 + \bar{t}^\beta - \frac{\bar{h}}{2} \left[\frac{\bar{h}}{2} (\dot{p}^\beta + \ddot{p}^\beta) - \dot{c}^\beta + \ddot{c}^\beta \right] \right\}. \quad (5.118) \end{aligned}$$

The equations for the stress boundary conditions on the top and bottom surfaces are the same as Eqs. (5.74) and (5.75) of Sect. 5.2.2. However, recall that the δe^{33} quantities have been redefined per Eqs. (5.112).

The displacement boundary conditions are given by Eqs. (5.82).

5.3.3. Variation of strain

By reducing the equations of Sect. 5.2.3, the strain variations are as follows:

$$\delta \Gamma'_{\alpha\beta} = \frac{1}{2} \left[\delta U_{\alpha/\beta} + \delta U_{\beta/\alpha} + \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\alpha/\beta} + \delta \bar{\Psi}_{\beta/\alpha} + 2 \bar{U}_{3,\alpha\beta} + \bar{h} \delta \bar{\Psi}_{3,\alpha\beta}) - 2 \bar{\Theta}^3 (\delta \bar{U}_{3,\alpha\beta} + \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha\beta}) \right]; \quad (5.119)$$

$$\delta \bar{\Gamma}_{\alpha\beta} = \frac{1}{2} \left[\delta \bar{U}_{\alpha/\beta} + \delta \bar{U}_{\beta/\alpha} + \bar{\Theta}^3 (\delta \bar{\Psi}_{\alpha/\beta} + \delta \bar{\Psi}_{\beta/\alpha}) \right]; \quad (5.120)$$

$$\delta \bar{\Gamma}_{\alpha 3} = \frac{1}{2} (\delta \bar{\Psi}_{\alpha} + \delta \bar{U}_{3,\alpha} + \bar{\Theta}^3 \delta \bar{\Psi}_{3,\alpha}); \quad (5.121)$$

$$\delta \bar{\Gamma}_{33} = \delta \bar{\Psi}_3; \quad (5.122)$$

$$\delta \Gamma''_{\alpha\beta} = \frac{1}{2} \left[\delta \bar{U}_{\alpha/\beta} + \delta \bar{U}_{\beta/\alpha} - \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\alpha/\beta} + \delta \bar{\Psi}_{\beta/\alpha} + 2 \bar{U}_{3,\alpha\beta} + \bar{h} \delta \bar{\Psi}_{3,\alpha\beta}) - 2 \bar{\Theta}^3 (\delta \bar{U}_{3,\alpha\beta} - \frac{\bar{h}}{2} \delta \bar{\Psi}_{3,\alpha\beta}) \right]. \quad (5.123)$$

5.3.4 Variation of stress

No change occurs; the stress variations are given by Eqs. (5.95), (5.106) and (5.107).

6. NUMERICAL EXAMPLE

Let us now consider a simple but practical problem, namely, to determine the critical buckling load of a simply supported rectangular sandwich plate subjected to a uniaxial compressive load of sufficient magnitude to induce stresses beyond the proportional limit. The plate is shown in Fig. 3.

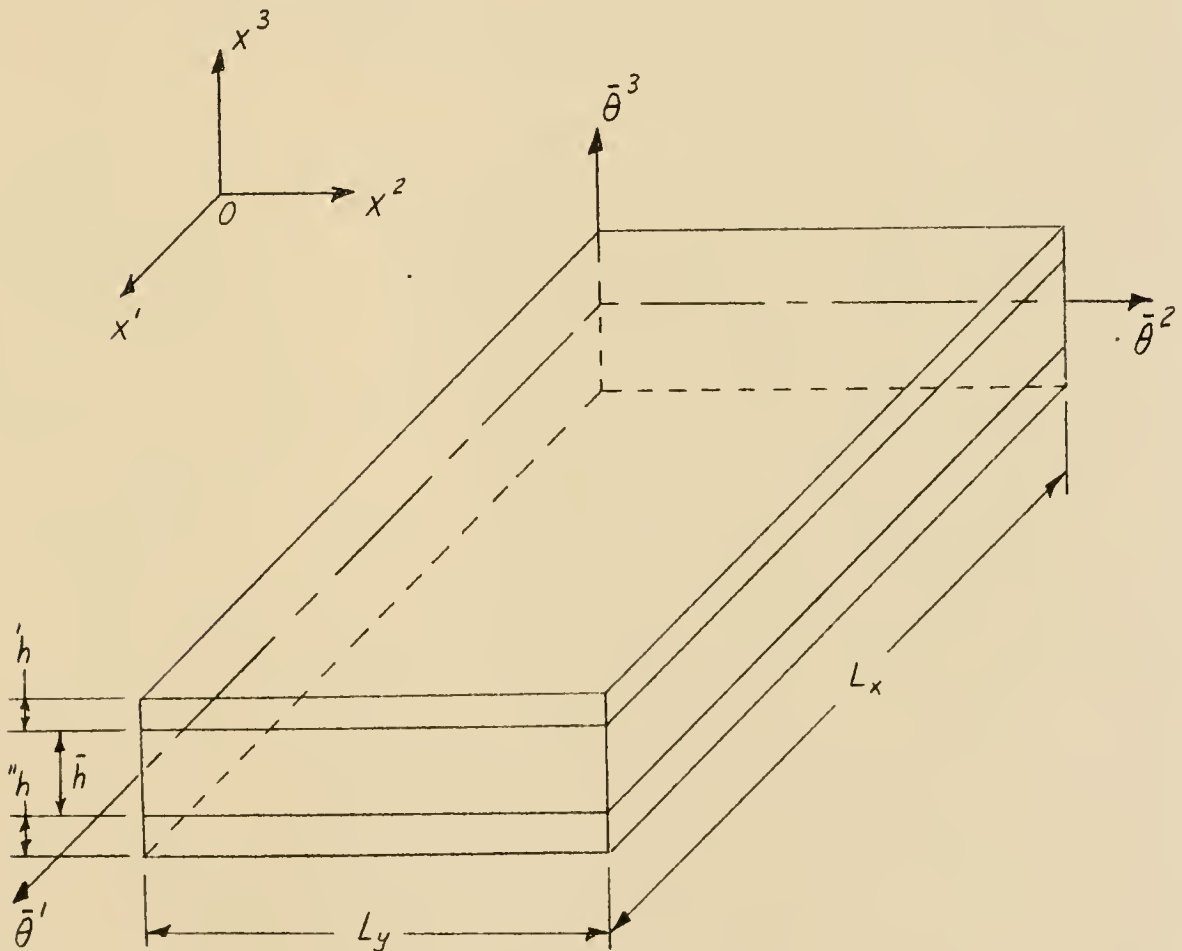


Fig. 3. Rectangular sandwich plate

Because of the geometry, the curvilinear coordinate system becomes a rectangular Cartesian coordinate system. Therefore, all contravariant tensors become Cartesian tensors, covariant derivatives become partial derivatives, metric tensors become Kronecker deltas, etc.

To obtain a solution, the equations of Sect. 5 will be employed; in particular, the modified buckling equations of Sect. 5.3. In addition to the assumptions of Sect. 5, the analysis will initially be based on the following two assumptions:

1. The face layers are very thin, so thin that they have no resistance to bending.
2. The middle layer is a so-called soft core; that is, the only significant core stresses are the transverse shear stresses.

The first assumption requires that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{S}_{\alpha\beta} \dot{\theta}_3 d\dot{\theta}_3 = 0, \quad (6.1)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \ddot{S}_{\alpha\beta} \ddot{\theta}_3 d\ddot{\theta}_3 = 0. \quad (6.2)$$

Recalling Eq. (3.3), i.e.,

$$\dot{\theta}_3 = \bar{\theta}_3 - \frac{1}{2} (\bar{h} + \dot{h})$$

Eq. (6.1) can be written as

$$\int_b^a \dot{S}_{\alpha\beta} \left[\bar{\theta}_3 - \frac{1}{2} (\bar{h} + \dot{h}) \right] d\bar{\theta}_3 = 0.$$

From the definitions of Eqs. (3.28), the above becomes

$$\dot{M}_{\alpha\beta} = \frac{1}{2} (\bar{h} + \dot{h}) \dot{N}_{\alpha\beta}. \quad (6.3)$$

Similarly, through Eqs. (3.2) and (3.28), Eq. (6.2) can be written as

$$\ddot{M}_{\alpha\beta} = -\frac{1}{2} (\bar{h} + \ddot{h}) \ddot{N}_{\alpha\beta}. \quad (6.4)$$

Obviously, equivalent relations hold for the variational stress resultants; then,

$$\delta \dot{m}_{\alpha\beta} = \frac{1}{2} (\bar{h} + \dot{h}) \delta \dot{n}_{\alpha\beta}, \quad (6.5)$$

$$\delta \ddot{m}_{\alpha\beta} = -\frac{1}{2} (\bar{h} + \ddot{h}) \delta \ddot{n}_{\alpha\beta}. \quad (6.6)$$

The second assumption requires that

$$\bar{S}_{33} = \bar{S}_{\alpha\beta} = 0, \quad (6.7)$$

which implies that

$$\delta \bar{x}_{33} = \delta \bar{x}_{\alpha\beta} = 0. \quad (6.8)$$

Therefore, all the stress resultants that depend upon the above stresses are also zero, e.g.,

$$\bar{N}_{\alpha\beta} = \bar{M}_{\alpha\beta} = \bar{N}_{33} = \delta \bar{m}_{\alpha\beta} = \delta \bar{m}_{\alpha\beta} = \delta \bar{m}_{33} = 0. \quad (6.9)$$

Also, in Sect. 5.1.4, it was found that under the premise of

$$\dot{\gamma}_{i3} = 0$$

the variational principle yielded the conclusion that

$$\dot{S}_{i3} = \delta \dot{\Gamma}_{i3} = \delta \dot{\mathcal{L}}_{i3} = 0.$$

Similarly, the assumption of Eq. (6.7) implies not only Eq. (6.8)

but also

$$\bar{\gamma}_{33} = \bar{\gamma}_{\alpha\beta} = \delta \bar{\Gamma}_{33} = \delta \bar{\Gamma}_{\alpha\beta} = 0. \quad (6.10)$$

When substituted into Eqs. (5.46) and (5.92), the above yields

$$\bar{\psi}_3 = \bar{\varphi}_3 = 0. \quad (6.11)$$

When $\bar{\psi}_3 = 0$ is substituted into Eqs. (5.4), we obtain

$$\dot{\psi}_\alpha = \ddot{\psi}_\alpha = -\bar{u}_{3,\alpha}; \quad (6.12)$$

and, thus, the equations relating the displacement functions, corresponding to Eqs. (5.6) and (5.7), are as follows:

$$\begin{aligned} \dot{u}_\alpha &= \bar{u}_\alpha + \frac{\bar{h}}{2} \bar{\psi}_\alpha - \frac{\bar{h}}{2} \bar{u}_{3,\alpha}, & \dot{u}_3 &= \bar{u}_3, \\ \ddot{u}_\alpha &= \bar{u}_\alpha - \frac{\bar{h}}{2} \bar{\psi}_\alpha + \frac{\bar{h}}{2} \bar{u}_{3,\alpha}, & \ddot{u}_3 &= \bar{u}_3; \end{aligned} \quad (6.13)$$

$$\begin{aligned}
 \bar{u}_\alpha &= \frac{1}{2} \left[\dot{u}_\alpha + \ddot{u}_\alpha + \frac{1}{2} (\dot{h} - \ddot{h}) \dot{u}_{3,\alpha} \right], & \bar{u}_3 &= \frac{1}{2} (\dot{u}_3 + \ddot{u}_3), \\
 \bar{\psi}_\alpha &= \frac{1}{h} \left[\dot{u}_\alpha - \ddot{u}_\alpha + \frac{1}{2} (\dot{h} + \ddot{h}) \dot{u}_{3,\alpha} \right], & \bar{\psi}_3 &= 0.
 \end{aligned}
 \tag{6.14}$$

The equations for the displacements of the three layers become

$$\begin{aligned}
 \dot{v}_\alpha &= \bar{u}_\alpha + \frac{\bar{h}}{2} (\bar{\psi}_\alpha + \bar{u}_{3,\alpha}) - \bar{\theta}_3 \bar{u}_{3,\alpha}, & \dot{v}_3 &= \bar{u}_3, \\
 \bar{v}_\alpha &= \bar{u}_\alpha + \bar{\theta}_3 \bar{\psi}_\alpha, & \bar{v}_3 &= \bar{u}_3, \\
 \ddot{v}_\alpha &= \bar{u}_\alpha - \frac{\bar{h}}{2} (\bar{\psi}_\alpha + \bar{u}_{3,\alpha}) - \bar{\theta}_3 \bar{u}_{3,\alpha}, & \ddot{v}_3 &= \bar{u}_3.
 \end{aligned}
 \tag{6.15}$$

Similarly,

$$\begin{aligned}
 \dot{\Psi}_\alpha &= \ddot{\Psi}_\alpha = - \bar{U}_{3,\alpha}, \\
 \dot{U}_\alpha &= \bar{U}_\alpha + \frac{\bar{h}}{2} \bar{\Psi}_\alpha - \frac{\dot{h}}{2} \bar{U}_{3,\alpha}, & \dot{U}_3 &= \bar{U}_3, \\
 \ddot{U}_\alpha &= \bar{U}_\alpha - \frac{\bar{h}}{2} \bar{\Psi}_\alpha + \frac{\ddot{h}}{2} \bar{U}_{3,\alpha}, & \ddot{U}_3 &= \bar{U}_3, \\
 \bar{U}_\alpha &= \frac{1}{2} \left[\dot{U}_\alpha + \ddot{U}_\alpha + \frac{1}{2} (\dot{h} - \ddot{h}) \dot{U}_{3,\alpha} \right], & \bar{U}_3 &= \frac{1}{2} (\dot{U}_3 + \ddot{U}_3), \\
 \bar{\Psi}_\alpha &= \frac{1}{h} \left[\dot{U}_\alpha - \ddot{U}_\alpha + \frac{1}{2} (\dot{h} + \ddot{h}) \dot{U}_{3,\alpha} \right], & \bar{\Psi}_3 &= 0;
 \end{aligned}
 \tag{6.16}$$

and

$$\begin{aligned}
V'_\alpha &= \bar{U}_\alpha + \frac{\bar{h}}{2} (\bar{\Psi}_\alpha + \bar{U}_{3,\alpha}) - \bar{\theta}_3 \bar{U}_{3,\alpha}, & V'_3 &= \bar{U}_3, \\
\bar{V}_\alpha &= \bar{U}_\alpha + \bar{\theta}_3 \bar{\Psi}_\alpha, & \bar{V}_3 &= \bar{U}_3, \\
V''_\alpha &= \bar{U}_\alpha - \frac{\bar{h}}{2} (\bar{\Psi}_\alpha + \bar{U}_{3,\alpha}) - \bar{\theta}_3 \bar{U}_{3,\alpha}, & V''_3 &= \bar{U}_3.
\end{aligned} \tag{6.17}$$

6.1. Field Equations

In Sect. 5.3.1, four field equations were established; however, of this set, Eq. (5.111) was a result of the arbitrary quantity $\delta\bar{\Psi}_3$ in the variational equation. Because $\bar{\Psi}_3$ is zero for this problem, Eq. (5.111) must be omitted. Then, with Eqs. (6.5), (6.6), (6.9) and (6.11) substituted into Eqs. (5.108) - (5.110), the field equations become

$$(\dot{m}_{\alpha\beta} + \ddot{m}_{\alpha\beta})_{,\beta} + \dot{p}_\alpha + \bar{p}_\alpha + \ddot{p}_\alpha = 0, \tag{6.18}$$

$$\frac{\bar{h}}{2} (\dot{m}_{\alpha\beta} - \ddot{m}_{\alpha\beta})_{,\beta} - \bar{g}_\alpha + \frac{\bar{h}}{2} (\dot{p}_\alpha - \ddot{p}_\alpha) + \bar{c}_\alpha = 0, \tag{6.19}$$

$$\begin{aligned}
& [(\dot{N}_{\alpha\beta} + \ddot{N}_{\alpha\beta}) \bar{U}_{3,\alpha} + \frac{1}{2} (\dot{h} \dot{m}_{\alpha\beta} - \ddot{h} \ddot{m}_{\alpha\beta})_{,\alpha} + \bar{g}_\beta \\
& - \frac{\bar{h}}{2} (\dot{p}_\beta - \ddot{p}_\beta) + \dot{c}_\beta + \ddot{c}_\beta]_{,\beta} + \dot{p}_3 + \bar{p}_3 + \ddot{p}_3 = 0.
\end{aligned} \tag{6.20}$$

Next, by assuming that the variation of body forces are negligible and by applying stress continuity at the interfaces, the definitions of Eqs. (3.119) and (5.70) along with Eqs. (5.112) and (6.11) yield

$$\begin{aligned} {}_1P_i + \bar{P}_i + {}_2P_i &= {}_2P_i - {}_1P_i, \\ \frac{\bar{h}}{2} ({}_1P_\alpha - {}_2P_\alpha) + \bar{C}_\alpha &= \frac{\bar{h}}{2} ({}_2P_\alpha + {}_1P_\alpha), \\ \frac{\bar{h}}{2} ({}_1P_\alpha - {}_2P_\alpha) - \bar{C}_\alpha - {}_2C_\alpha &= -h {}_2P_\alpha - {}_1P_\alpha. \end{aligned} \quad (6.21)$$

With the above and by using the first two field equations to simplify the third equation, the field equations become

$${}_1m_{\alpha\beta,\beta} + {}_2m_{\alpha\beta,\beta} + {}_2P_\alpha - {}_1P_\alpha = 0, \quad (6.22)$$

$$\frac{\bar{h}}{2} ({}_1m_{\alpha\beta,\beta} - {}_2m_{\alpha\beta,\beta}) - \bar{g}_\alpha + \frac{\bar{h}}{2} ({}_2P_\alpha + {}_1P_\alpha) = 0, \quad (6.23)$$

$$\begin{aligned} ({}_1N_{\alpha\beta} + {}_2N_{\alpha\beta}) \bar{U}_{3,\alpha\beta} + \hat{h} \bar{g}_{\alpha,\alpha} + \frac{1}{2} (h {}_2P_\alpha + {}_1P_\alpha)_{,\alpha} \\ + {}_2P_3 - {}_1P_3 = 0, \end{aligned} \quad (6.24)$$

where

$$\hat{h} = \frac{1}{\bar{h}} \left(\bar{h} + \frac{h + {}_1h}{2} \right). \quad (6.25)$$

6.2. Boundary Conditions

The basic equations for the boundary conditions were established in Sect. 5.3.2. As mentioned in the previous section, the fact that $\delta\bar{\Psi}_3$ does not exist for this case requires the omission of certain equations, namely, Eqs. (5.116) and (5.118). Substitution of Eqs. (6.5), (6.6), (6.9), (6.11) and (6.21) into the remaining equations yields the stress boundary conditions as follows:

on the top surface,

$${}_2P_{*i} - {}_2P_i = 0, \quad {}_2C_{*i} - {}_2C_i = 0; \quad (6.26)$$

on the bottom surface,

$${}_1P_{*i} + {}_1P_i = 0, \quad {}_1C_{*i} + {}_1C_i = 0; \quad (6.27)$$

on the edge surface,

$${}_0\dot{s}_\alpha + {}_0\bar{s}_\alpha + {}_0''s_\alpha = {}_0\bar{m}_\beta (\dot{m}_{\alpha\beta} + ''m_{\alpha\beta}), \quad (6.28)$$

$$\frac{\bar{h}}{2}({}_0\dot{s}_\alpha - {}_0''s_\alpha) + {}_0\bar{\ell}_\alpha = {}_0\bar{m}_\beta (\dot{m}_{\alpha\beta} - ''m_{\alpha\beta}) \frac{\bar{h}}{2}, \quad (6.29)$$

$${}_0\dot{\ell}_\alpha + {}_0''\ell_\alpha - \frac{\bar{h}}{2}({}_0\dot{s}_\alpha - {}_0''s_\alpha) = \frac{1}{2} {}_0\bar{m}_\beta (\dot{h} \dot{m}_{\alpha\beta} - \dot{h} ''m_{\alpha\beta}), \quad (6.30)$$

$$\begin{aligned} {}_0\dot{s}_3 + {}_0\bar{s}_3 + {}_0''s_3 = & {}_0\bar{m}_\beta [(\dot{N}_{\alpha\beta} + ''N_{\alpha\beta})\bar{U}_{3,\alpha} + \bar{g}_{\beta} \\ & + \frac{1}{2}(\dot{h} \dot{m}_{\alpha\beta} - \dot{h} ''m_{\alpha\beta})_{,\alpha} + \dot{h} {}_2P_\beta + \dot{h} {}_1P_\beta]. \end{aligned} \quad (6.31)$$

The displacement boundary conditions are as follows:

on the top surface,

$$V_i - \tilde{V}_i = 0; \quad (6.32)$$

on the bottom surface,

$$V_i - \tilde{V}_i = 0; \quad (6.33)$$

on the edge surface,

$$\begin{aligned} V_i - \tilde{V}_i &= 0, \\ \bar{V}_i - \tilde{\bar{V}}_i &= 0, \\ V_i - \tilde{V}_i &= 0. \end{aligned} \quad (6.34)$$

6.3. Variation of Strain

With Eqs. (6.10) and (6.11), the strain variation equations of Sect. 5.3.3 reduce to the following:

$$\begin{aligned} \delta \Gamma_{\alpha\beta} &= \frac{1}{2} \left[\delta U_{\alpha,\beta} + \delta U_{\beta,\alpha} + \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\alpha,\beta} + \delta \bar{\Psi}_{\beta,\alpha} \right. \\ &\quad \left. + 2 \bar{U}_{3,\alpha\beta}) - 2 \bar{\Theta}_3 \delta \bar{U}_{3,\alpha\beta} \right]; \end{aligned} \quad (6.35)$$

$$\delta \bar{\Gamma}_{\alpha 3} = \frac{1}{2} (\delta \bar{\Psi}_{\alpha} + \delta \bar{U}_{3, \alpha}) ; \quad (6.36)$$

$$\begin{aligned} \delta \bar{\Gamma}_{\alpha \beta} = \frac{1}{2} [& \delta \bar{U}_{\alpha, \beta} + \delta U_{\beta, \alpha} - \frac{h}{2} (\delta \bar{\Psi}_{\alpha, \beta} + \delta \bar{\Psi}_{\beta, \alpha} \\ & + 2 \delta U_{3, \alpha \beta}) - 2 \bar{\theta}_3 \delta \bar{U}_{3, \alpha \beta}] . \end{aligned} \quad (6.37)$$

6.4. Variation of Stress

With reference to Sect. 5.3.4 and with Eq. (6.8), the stress variation relations are given by Eqs. (5.95), (5.106) and (5.107); by substituting into these equations the strain variations of Sect. 6.3, we obtain

$$\begin{aligned} \delta \bar{s}_{\alpha \beta} = \frac{1}{2} \bar{R}_{\alpha \beta \gamma \delta} [& \delta \bar{U}_{\gamma, \delta} + \delta \bar{U}_{\delta, \gamma} + \frac{h}{2} (\delta \bar{\Psi}_{\gamma, \delta} + \delta \bar{\Psi}_{\delta, \gamma} \\ & + 2 \bar{U}_{3, \gamma \delta}) - 2 \bar{\theta}_3 \delta \bar{U}_{3, \gamma \delta}] , \end{aligned}$$

$$\delta \bar{s}_{\alpha 3} = \bar{M}_{\alpha 3 \gamma 3} (\delta \bar{\Psi}_{\gamma} + \delta U_{3, \gamma}) , \quad (6.38)$$

$$\begin{aligned} \delta \bar{s}_{\alpha \beta} = \frac{1}{2} \bar{R}_{\alpha \beta \gamma \delta} [& \delta \bar{U}_{\gamma, \delta} + \delta \bar{U}_{\delta, \gamma} - \frac{h}{2} (\delta \bar{\Psi}_{\gamma, \delta} + \delta \bar{\Psi}_{\delta, \gamma} \\ & + 2 \bar{U}_{3, \gamma \delta}) - 2 \bar{\theta}_3 \delta \bar{U}_{3, \gamma \delta}] . \end{aligned}$$

Because of the symmetry relations of Eq. (5.105), we have

$$'R_{\alpha\beta\gamma\delta} \delta \bar{U}_{\gamma,\delta} = 'R_{\alpha\beta\delta\gamma} \delta \bar{U}_{\delta,\gamma}, \text{ etc.}$$

Therefore, the stress variations can be written more compactly as follows:

$$\delta 'N_{\alpha\beta} = 'R_{\alpha\beta\gamma\delta} \left[\delta \bar{U}_{\gamma,\delta} + \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\gamma,\delta} + \delta \bar{U}_{3,\gamma\delta}) - \bar{\Theta}_3 \delta \bar{U}_{3,\gamma\delta} \right], \quad (6.39)$$

$$\delta \bar{S}_{\alpha 3} = \bar{M}_{\alpha 3\gamma 3} (\delta \bar{\Psi}_{\gamma} + \delta \bar{U}_{3,\gamma}), \quad (6.40)$$

$$\delta ''N_{\alpha\beta} = ''R_{\alpha\beta\gamma\delta} \left[\delta \bar{U}_{\gamma,\delta} - \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\gamma,\delta} + \delta \bar{U}_{3,\gamma\delta}) - \bar{\Theta}_3 \delta \bar{U}_{3,\gamma\delta} \right]. \quad (6.41)$$

6.5. Solution

As previously mentioned, the purpose of the modified buckling equations is to avoid solving the prebuckling problem. However, for a formal solution, the initial state of stress in the sandwich plate should be determined from the prebuckling stress boundary conditions; in particular, Eq. (5.33) which reduces to

$${}_0\dot{S}_{\alpha} + {}_0\bar{S}_{\alpha} + {}_0''S_{\alpha} = {}_0m_{\beta} ({}'_N_{\alpha\beta} + {}''N_{\alpha\beta}) \quad (6.42)$$

for this case. If the uniaxial compressive load, say $-R_2$, is taken to act in the θ_2 direction; then, Eq. (6.42) yields

$$\dot{N}_{22} + \ddot{N}_{22} = -R_2, \quad (6.43)$$

$$\dot{N}_{11} + \ddot{N}_{11} = \dot{N}_{12} + \ddot{N}_{12} = 0. \quad (6.44)$$

During buckling, the upper and lower surfaces are free of tractions, i.e.,

$$\delta_2 \dot{P}_{*i} = \delta_2 \ddot{P}_{*i} = 0;$$

this is a boundary condition. Therefore, Eqs. (6.26) and (6.27) yield

$$\dot{P}_i = \ddot{P}_i = 0. \quad (6.45)$$

The remaining surfaces for which boundary conditions must be specified are the edges. Because the edges are simply supported, it is known that

$$\delta \tilde{V}_3 = \delta \bar{\tilde{V}}_3 = \delta \ddot{\tilde{V}}_3 = 0$$

which from Eqs. (6.34) give

$$\dot{V}_3 = \bar{V}_3 = \ddot{V}_3 = 0. \quad (6.46)$$

As for the other displacement components on the edges, we only know that a motion in the θ_2 direction must occur at the edges

$\theta_2 = 0$, L_y . All other displacements may or may not be restrained; fortunately, we can leave this decision unresolved for now.

From Eqs. (6.43) - (6.45), the field equations become

$$'m_{\alpha\beta,\beta} + ''m_{\alpha\beta,\beta} = 0, \quad (6.47)$$

$$\frac{\bar{h}}{2} ('m_{\alpha\beta,\beta} - ''m_{\alpha\beta,\beta}) - \bar{f}_\alpha = 0, \quad (6.48)$$

$$-R_2 \bar{U}_{3,22} + \hat{h} \bar{f}_{\alpha,\alpha} = 0. \quad (6.49)$$

Recalling the definitions of the stress resultant variations, e.g.,

$$\delta 'm_{\alpha\beta} = \int_b^a \delta 's_{\alpha\beta} d\bar{\theta}_3,$$

the stress variations, Eqs. (6.39), can be applied to obtain

$$\begin{aligned} \delta 'm_{\alpha\beta} = \int_b^a R_{\alpha\beta\gamma\delta} [\delta \bar{U}_{\gamma,\delta} + \frac{\bar{h}}{2} (\delta \bar{\Psi}_{\gamma,\delta} + \delta \bar{U}_{3,\gamma\delta}) \\ - \bar{\theta}_3 \delta \bar{U}_{3,\gamma\delta}] d\bar{\theta}_3. \end{aligned}$$

Integration yields

$$'m_{\alpha\beta} = 'h R_{\alpha\beta\gamma\delta} (\bar{U}_{\gamma,\delta} + \frac{\bar{h}}{2} \bar{\Psi}_{\gamma,\delta} - \frac{\bar{h}}{2} \bar{U}_{3,\gamma\delta}). \quad (6.50)$$

Similarly,

$$''m_{\alpha\beta} = ''h R_{\alpha\beta\gamma\delta} (\bar{U}_{\gamma,\delta} - \frac{\bar{h}}{2} \bar{\Psi}_{\gamma,\delta} + \frac{\bar{h}}{2} \bar{U}_{3,\gamma\delta}), \quad (6.51)$$

$$\bar{g}_\alpha = \bar{h} \bar{M}_{\alpha 3 \gamma 3} (\bar{\Psi}_\gamma + \bar{U}_{3, \gamma}). \quad (6.52)$$

After substitution of Eqs. (6.50) - (6.52) into Eqs. (6.47) - (6.49), the equations become

$$\begin{aligned} & \hat{h} \hat{R}_{\alpha \beta \gamma \delta} (\bar{U}_{\gamma, \delta \beta} + \frac{\bar{h}}{2} \bar{\Psi}_{\gamma, \delta \beta} - \frac{\bar{h}}{2} \bar{U}_{3, \gamma \delta \beta}) \\ & + \hat{h} \hat{R}_{\alpha \beta \gamma \delta} (\bar{U}_{\gamma, \delta \beta} - \frac{\bar{h}}{2} \bar{\Psi}_{\gamma, \delta \beta} + \frac{\bar{h}}{2} \bar{U}_{3, \gamma \delta \beta}) = 0, \end{aligned} \quad (6.53)$$

$$\begin{aligned} & \hat{h} \hat{R}_{\alpha \beta \gamma \delta} (\bar{U}_{\gamma, \delta \beta} + \frac{\bar{h}}{2} \bar{\Psi}_{\gamma, \delta \beta} - \frac{\bar{h}}{2} \bar{U}_{3, \gamma \delta \beta}) \\ & - \hat{h} \hat{R}_{\alpha \beta \gamma \delta} (\bar{U}_{\gamma, \delta \beta} - \frac{\bar{h}}{2} \bar{\Psi}_{\gamma, \delta \beta} + \frac{\bar{h}}{2} \bar{U}_{3, \gamma \delta \beta}) \\ & - 2 \bar{M}_{\alpha 3 \gamma 3} (\bar{\Psi}_\gamma + \bar{U}_{3, \gamma}) = 0, \end{aligned} \quad (6.54)$$

$$- R_2 \bar{U}_{3, 22} + \hat{h} \bar{h} \bar{M}_{\alpha 3 \gamma 3} (\bar{\Psi}_{\gamma, \alpha} + \bar{U}_{3, \gamma \alpha}) = 0. \quad (6.55)$$

To obtain a solution to the above equations, concepts employed in the work of Chang, Ebcioğlu, and Baltes [26] will be applied.

It may be recalled that in Sect. 3.2 it was mentioned that the basic equations could be formulated in terms of two sets of displacement functions. Examination of Eqs. (6.16) shows that for this problem the field equations can be simplified by converting to the alternate set of displacement functions. Thus, substitution of Eqs. (6.16) into Eqs. (6.53) - (6.55) gives

$$\dot{h} \dot{R}_{\alpha\beta\gamma\delta} \dot{U}_{\gamma,\delta\beta} + \ddot{h} \ddot{R}_{\alpha\beta\gamma\delta} \ddot{U}_{\gamma,\delta\beta} = 0, \quad (6.56)$$

$$\begin{aligned} \dot{h} \dot{R}_{\alpha\beta\gamma\delta} \dot{U}_{\gamma,\delta\beta} - \ddot{h} \ddot{R}_{\alpha\beta\gamma\delta} \ddot{U}_{\gamma,\delta\beta} \\ - 2 \bar{M}_{\alpha 3 \gamma 3} \left[\frac{1}{\dot{h}} (\dot{U}_{\gamma} - \ddot{U}_{\gamma}) + \hat{h} \dot{U}_{3,\gamma} \right] = 0, \end{aligned} \quad (6.57)$$

$$- R_2 \dot{U}_{3,22} + \hat{h} \bar{h} \bar{M}_{\alpha 3 \gamma 3} \left[\frac{1}{\dot{h}} (\dot{U}_{\gamma,\alpha} - \ddot{U}_{\gamma,\alpha}) + \hat{h} \dot{U}_{3,\gamma\alpha} \right] = 0. \quad (6.58)$$

Eq. (6.56) is identically satisfied if

$$\dot{h} \dot{R}_{\alpha\beta\gamma\delta} = \omega \ddot{h} \ddot{R}_{\alpha\beta\gamma\delta}, \quad (6.59)$$

$$\ddot{U}_{\gamma} = -\omega \dot{U}_{\gamma}, \quad (6.60)$$

where ω is a constant. Therefore, substitution of Eqs. (6.59) and (6.60) into Eqs. (6.56) - (6.58) yields

$$0 = 0,$$

$$\dot{h} \dot{R}_{\alpha\beta\gamma\delta} \dot{U}_{\gamma,\delta\beta} - \bar{M}_{\alpha 3 \gamma 3} \left(\frac{1+\omega}{\dot{h}} \dot{U}_{\gamma} + \hat{h} \dot{U}_{3,\gamma} \right) = 0, \quad (6.61)$$

$$- R_2 \dot{U}_{3,22} + \hat{h} \bar{h} \bar{M}_{\alpha 3 \gamma 3} \left(\frac{1+\omega}{\dot{h}} \dot{U}_{\gamma,\alpha} + \hat{h} \dot{U}_{3,\gamma\alpha} \right) = 0. \quad (6.62)$$

It has been established that the only significant prebuckling stresses are

$$\dot{S}_{22}, \dot{S}_{22}, \bar{S}_{13}, \bar{S}_{23}.$$

Then, Eqs. (5.101) - (5.103) show that

$$\dot{R}_{1112} = \dot{R}_{1121} = \dot{R}_{1211} = \dot{R}_{2111} = 0,$$

$$\dot{R}_{1222} = \dot{R}_{2122} = \dot{R}_{2212} = \dot{R}_{2221} = 0;$$

therefore, when expanded, Eqs. (6.61) and (6.62) become

$$\begin{aligned} & \dot{h} [\dot{R}_{1111} \dot{U}_{1,11} + \dot{R}_{1212} \dot{U}_{1,22} + (\dot{R}_{1122} + \dot{R}_{1212}) \dot{U}_{2,12}] \\ & - \bar{M}_{1313} \left(\frac{1+\omega}{\bar{h}} \dot{U}_1 + \hat{h} \dot{U}_{3,1} \right) - \bar{M}_{1323} \left(\frac{1+\omega}{\bar{h}} \dot{U}_2 + \hat{h} \dot{U}_{3,2} \right) = 0, \end{aligned} \quad (6.63)$$

$$\begin{aligned} & \dot{h} [(\dot{R}_{1122} + \dot{R}_{1212}) \dot{U}_{1,12} + \dot{R}_{1212} \dot{U}_{2,11} + \dot{R}_{2222} \dot{U}_{2,22}] \\ & - \bar{M}_{1323} \left(\frac{1+\omega}{\bar{h}} \dot{U}_1 + \hat{h} \dot{U}_{3,1} \right) - \bar{M}_{2323} \left(\frac{1+\omega}{\bar{h}} \dot{U}_2 + \hat{h} \dot{U}_{3,2} \right) = 0, \end{aligned} \quad (6.64)$$

$$\begin{aligned} & \hat{h} (1+\omega) [\bar{M}_{1313} \dot{U}_{1,1} + \bar{M}_{1323} (\dot{U}_{1,2} + \dot{U}_{2,1}) + \bar{M}_{2323} \dot{U}_{2,2}] \\ & + \hat{h}^2 \bar{h} (\bar{M}_{1313} \dot{U}_{3,11} + 2 \bar{M}_{1323} \dot{U}_{3,12}) \\ & + (\hat{h}^2 \bar{h} \bar{M}_{2323} - R_2) \dot{U}_{3,22} = 0, \end{aligned} \quad (6.65)$$

where (the primes and bars are omitted for convenience)

$$M_{1313} = \frac{S_o}{Y_o} + \frac{2}{3 S_o^2} \left(\frac{\delta S_o}{\delta Y_o} - \frac{S_o}{Y_o} \right) (S_{13})^2, \quad (6.66)$$

$$M_{1323} = \frac{2}{3 S_o^2} \left(\frac{\delta S_o}{\delta Y_o} - \frac{S_o}{Y_o} \right) S_{13} S_{23}, \quad (6.67)$$

$$M_{2323} = \frac{S_o}{Y_o} + \frac{2}{3 S_o^2} \left(\frac{\delta S_o}{\delta Y_o} - \frac{S_o}{Y_o} \right) (S_{23})^2, \quad (6.68)$$

$$R_{1111} = 2 \frac{S_o}{Y_o} + 2 \frac{S_o}{Y_o} \left(\frac{3k-2 \frac{S_o}{Y_o}}{3k+4 \frac{S_o}{Y_o}} \right) + \phi \left(2 \frac{S_o}{Y_o} S_{22} \right)^2, \quad (6.69)$$

$$R_{1122} = 2 \frac{S_o}{Y_o} \left(\frac{3k-2 \frac{S_o}{Y_o}}{3k+4 \frac{S_o}{Y_o}} \right) - \phi \left(2 \frac{S_o}{Y_o} \right) \left(3k+2 \frac{S_o}{Y_o} \right) (S_{22})^2, \quad (6.70)$$

$$R_{1212} = \frac{S_o}{Y_o} \quad (6.71)$$

$$R_{2222} = 2 \frac{S_o}{Y_o} + 2 \frac{S_o}{Y_o} \left(\frac{3k-2 \frac{S_o}{Y_o}}{3k+4 \frac{S_o}{Y_o}} \right) + \phi \left[\left(3k+2 \frac{S_o}{Y_o} \right) S_{22} \right]^2. \quad (6.72)$$

From Eqs. (3.91) and (5.104)

$$\dot{S}_o = \frac{\sqrt{2}}{3} \dot{S}_{22}, \quad (6.73)$$

$$\dot{\phi} = \frac{3 \left(\frac{\delta \dot{S}_o}{\delta \dot{Y}_o} - \frac{\dot{S}_o}{\dot{Y}_o} \right)}{\dot{S}_{22}^2 \left(3k+4 \frac{\dot{S}_o}{\dot{Y}_o} \right) \left(3k+3 \frac{\dot{S}_o}{\dot{Y}_o} + \frac{\delta \dot{S}_o}{\delta \dot{Y}_o} \right)}. \quad (6.74)$$

Although a simplified large strain theory is being employed, it is usually assumed in the literature that the stress and strain tensors can be interpreted in the classical sense. Then, the material constants of classical theory are applicable, e.g.,

$$G = \frac{E}{2(1+\mu)} \quad , \quad (6.75)$$

$$k_1 = \frac{E}{3(1-2\mu)} \quad , \quad (6.76)$$

where G is the shear modulus, E is Young's modulus, μ is Poisson's ratio, and k_1 is the bulk modulus. As shown by Teodosiadis, Langhaar, and Smith [22], the following relations hold:

$$\frac{S_0}{\gamma_0} = G_s = \frac{E_s}{3 - (1-2\mu) \frac{E_s}{E}} \quad ; \quad (6.77)$$

$$\frac{\delta S_0}{\delta \gamma_0} = G_t = \frac{E_t}{3 - (1-2\mu) \frac{E_t}{E}} \quad ; \quad (6.78)$$

where G_s and G_t are the secant and tangent shear moduli, respectively, and E_s and E_t are the secant and tangent moduli, respectively, as obtained from a simple compression stress-strain curve.

In 1950, Hoff [27] found the solution to the elastic buckling problem of a rectangular sandwich plate by assuming displacement functions as follows:

$$U_1 = Y_1 \cos \frac{q\pi\theta_1}{L_x} \sin \frac{m\pi\theta_2}{L_y} ;$$

$$U_2 = Y_2 \sin \frac{q\pi\theta_1}{L_x} \cos \frac{m\pi\theta_2}{L_y} ; \quad (6.79)^*$$

$$U_3 = Y_3 \sin \frac{q\pi\theta_1}{L_x} \sin \frac{m\pi\theta_2}{L_y} .$$

These same relations were used by Chang, Ebcioğlu, and Baltes [26] to solve the plastic buckling problem. The above displacements satisfy the simply supported conditions, i.e.,

$$U_3 = 0, \quad \text{AT } \theta_1 = \left\{ \begin{matrix} 0 \\ L_x \end{matrix} \right\} \text{ AND } \theta_2 = \left\{ \begin{matrix} 0 \\ L_y \end{matrix} \right\} ;$$

and also impose the limitations:

$$U_1 = 0, \quad \text{AT } \theta_2 = \left\{ \begin{matrix} 0 \\ L_y \end{matrix} \right\} ;$$

$$U_2 = 0, \quad \text{AT } \theta_1 = \left\{ \begin{matrix} 0 \\ L_x \end{matrix} \right\} .$$

Substitution of Eqs. (6.79) into Eqs. (6.63) - (6.65) gives

* Actually, q was taken to be one. Here, we allow for higher buckling modes.

$$\begin{aligned}
& \left\{ \left[\hbar \dot{R}_{1111} \left(\frac{q\pi}{L_x} \right)^2 + \hbar \dot{R}_{1212} \left(\frac{m\pi}{L_y} \right)^2 + \frac{1+\omega}{\hbar} \bar{M}_{1313} \right] Y_1 \right. \\
& + \hbar (\dot{R}_{1122} + \dot{R}_{1212}) \left(\frac{q\pi}{L_x} \right) \left(\frac{m\pi}{L_y} \right) Y_2 \\
& + \hat{h} \left(\frac{q\pi}{L_x} \right) \bar{M}_{1313} Y_3 \left. \right\} \cos \frac{q\pi\theta_1}{L_x} \sin \frac{m\pi\theta_2}{L_y} \\
& + \left[\frac{1+\omega}{\hbar} Y_2 + \hat{h} \left(\frac{m\pi}{L_y} \right) Y_3 \right] \bar{M}_{1323} \sin \frac{q\pi\theta_1}{L_x} \cos \frac{m\pi\theta_2}{L_y} = 0,
\end{aligned} \tag{6.80}$$

$$\begin{aligned}
& \left\{ \hbar (\dot{R}_{1212} + \dot{R}_{1122}) \left(\frac{q\pi}{L_x} \right) \left(\frac{m\pi}{L_y} \right) Y_1 \right. \\
& + \left[\hbar \dot{R}_{1212} \left(\frac{q\pi}{L_x} \right)^2 + \hbar \dot{R}_{2222} \left(\frac{m\pi}{L_y} \right)^2 + \frac{1+\omega}{\hbar} \bar{M}_{2323} \right] Y_2 \\
& + \hat{h} \left(\frac{m\pi}{L_y} \right) \bar{M}_{2323} Y_3 \left. \right\} \sin \frac{q\pi\theta_1}{L_x} \cos \frac{m\pi\theta_2}{L_y} \\
& + \left[\frac{1+\omega}{\hbar} Y_1 + \hat{h} \left(\frac{q\pi}{L_x} \right) Y_3 \right] \bar{M}_{1323} \cos \frac{q\pi\theta_1}{L_x} \sin \frac{m\pi\theta_2}{L_y} = 0,
\end{aligned} \tag{6.81}$$

$$\begin{aligned}
& \left\{ \hat{h} (1+\omega) \left(\frac{q\pi}{L_x} \right) \bar{M}_{1313} Y_1 + \hat{h} (1+\omega) \left(\frac{m\pi}{L_y} \right) \bar{M}_{2323} Y_2 \right. \\
& + \left[\hat{h}^2 \bar{h} \left(\frac{q\pi}{L_x} \right)^2 \bar{M}_{1313} + (\hat{h}^2 \bar{h} \bar{M}_{2323} - R_2) \left(\frac{m\pi}{L_y} \right)^2 \right] Y_3 \left. \right\} \sin \frac{q\pi\theta_1}{L_x} \sin \frac{m\pi\theta_2}{L_y} \\
& + \left[\hat{h} (1+\omega) \left(\frac{m\pi}{L_y} \right) \bar{M}_{1323} Y_1 + \hat{h} (1+\omega) \left(\frac{q\pi}{L_x} \right) \bar{M}_{1323} Y_2 \right. \\
& + 2 \hat{h}^2 \bar{h} \left(\frac{q\pi}{L_x} \right) \left(\frac{m\pi}{L_y} \right) \bar{M}_{1323} Y_3 \left. \right] \cos \frac{q\pi\theta_1}{L_x} \cos \frac{m\pi\theta_2}{L_y} = 0.
\end{aligned} \tag{6.82}$$

If $\bar{M}_{1323}$ is zero, the equations can be written as

$$\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = 0$$

where

$$d_{11} = \dot{R}_{1111} q^2 + \dot{R}_{1212} \left(\frac{mL_x}{L_y} \right)^2 + \frac{\bar{G}(1+\omega)}{h\bar{h}} \left(\frac{L_x}{\pi} \right)^2,$$

$$d_{12} = d_{21} = (\dot{R}_{1122} + \dot{R}_{1212}) \frac{q mL_x}{L_y},$$

$$d_{13} = \frac{\bar{G}\hat{h}qL_x}{\pi h},$$

$$d_{22} = \dot{R}_{1212} q^2 + \dot{R}_{2222} \left(\frac{mL_x}{L_y} \right)^2 + \frac{\bar{G}(1+\omega)}{h\bar{h}} \left(\frac{L_x}{\pi} \right)^2,$$

$$d_{23} = \frac{\bar{G}\hat{h}mL_x^2}{\pi L_y},$$

$$d_{31} = \bar{G}\hat{h}(1+\omega) \left(\frac{qL_x}{\pi} \right),$$

$$d_{32} = \bar{G}\hat{h}(1+\omega) \left(\frac{mL_x^2}{\pi L_y} \right),$$

$$d_{33} = \bar{G}\bar{h}\hat{h}^2 q^2 + (\bar{G}\bar{h}\hat{h}^2 - R_2) \left(\frac{mL_x}{\pi} \right)^2,$$

and a unique solution exists if the determinant of the d_{ij} coefficients is zero. For $\bar{M}_{1323}$ to be zero, Eq. (6.67) requires that

$$\frac{\delta \bar{S}_0}{\delta \bar{\gamma}_0} = \frac{\bar{S}_0}{\bar{\gamma}_0}, \quad \text{OR} \quad \bar{G}_t = \bar{G}_s$$

which implies a Hookean material, which was assumed in reference [26].

Then,

$$\bar{G} = \bar{G}_s = \bar{G}_t ;$$

therefore,

$$\bar{M}_{1313} = \bar{M}_{2323} = \bar{G}.$$

The middle layer is elastic.

Because R_2 and the parameters $'R_{\alpha\beta\gamma\delta}$ depend upon the state of stress prior to buckling, an explicit expression for the critical buckling load cannot be obtained. However, parametric curves can be established; following the procedure of reference [26], the following parameters are introduced:

$$\begin{aligned} P_0 &= \left(\frac{\pi \hat{h} \bar{h}}{L_x} \right)^2 \frac{'h'E}{(1-\mu^2)(1+m)}, & m &= \frac{'h'E}{''h''E}; \\ H &= 4 \frac{S_0}{\gamma_0} \left(\frac{3k_1 + \frac{S_0}{\gamma_0}}{3k_1 + 4 \frac{S_0}{\gamma_0}} \right), & K_p &= \frac{(R_2)_{CR}}{P_0}, \\ \rho_1 &= \frac{\hat{h}}{2} (1+m), & \alpha &= \frac{\bar{h}}{L_x}, \\ \rho_2 &= \frac{\hat{h}}{2} (1+w), & \beta &= \frac{L_y}{mL_x}, \end{aligned} \quad (6.83a)$$

$$\begin{aligned}
\epsilon &= \frac{{}'E}{(1-'\mu^2){}'H} \quad , & r &= \frac{P_0}{\hat{h}^2 \bar{h} \bar{G}} \quad , \\
Z_{11} &= \frac{{}'R_{1111}}{{}'H} \quad , & Z_{12} &= \frac{{}'R_{1122}}{{}'H} \quad , & (6.83b) \\
Z_{22} &= \frac{{}'R_{2222}}{{}'H} \quad , & Z_{33} &= \frac{4 \, {}'R_{1212}}{{}'H} \quad ,
\end{aligned}$$

where $(R_2)_{cr}$ is the critical stress resultant. With the above parameters substituted into the equations for the d_{ij} coefficients, new coefficients are established as follows:

$$\beta_{11} = 2 Z_{11} q^2 + \frac{Z_{33}}{2 \beta^2} + \frac{2 \epsilon \rho_2}{r \rho_1} ;$$

$$\beta_{12} = \beta_{21} = \frac{2 q}{\beta} \left(Z_{12} + \frac{Z_{33}}{4} \right) ;$$

$$\beta_{13} = \frac{\pi \hat{h}^2 q \alpha \epsilon}{r \rho_1} ;$$

$$\beta_{22} = \frac{2 Z_{22}}{\beta^2} + \frac{Z_{33} q^2}{2} + \frac{2 \epsilon \rho_2}{r \rho_1} ;$$

$$\beta_{23} = \frac{\pi \hat{h}^2 \alpha \epsilon}{r \rho_1 \beta} ;$$

$$\beta_{31} = \frac{2 q \rho_2}{\pi r} ;$$

$$\beta_{32} = \frac{2 \rho_2}{\pi r \beta} ;$$

$$\beta_{33} = \left(\frac{q^2}{r} + \frac{1}{r \beta^2} - \frac{K_p}{\beta^2} \right) \hat{h}^2 \alpha .$$

Then, after some rather lengthy algebraic manipulations the equation

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{vmatrix} = 0$$

yields

$$\begin{aligned} K_p = \rho_1 \{ & (1 + g^2 \beta^2) \left[\frac{1}{2} (Z_{22} Z_{33} + 4 Z_{11} Z_{22} g^2 \beta^2 \right. \\ & - 2 Z_{12} Z_{33} g^2 \beta^2 - 4 Z_{12}^2 g^2 \beta^2 + Z_{11} Z_{33} g^4 \beta^4) r \rho_1 \\ & + 2 \beta^2 \epsilon \rho_2 (Z_{22} + Z_{11} g^2 \beta^2) \left. \right] - 2 g^2 \beta^4 \epsilon \rho_2 (Z_{11} \\ & + Z_{22} - Z_{33} - 2 Z_{12}) \} \times \{ r \rho_1 \left[\frac{1}{2} (Z_{22} Z_{33} \right. \\ & + 4 Z_{11} Z_{22} g^2 \beta^2 - 2 Z_{12} Z_{33} g^2 \beta^2 - 4 Z_{12}^2 g^2 \beta^2 \\ & + Z_{11} Z_{33} g^4 \beta^4) r \rho_1 + 2 \beta^2 \epsilon \rho_2 (Z_{22} + Z_{11} g^2 \beta^2) \left. \right] \\ & + \beta^2 \epsilon \rho_2 \left[\frac{1}{2} (1 + g^2 \beta^2) Z_{33} r \rho_1 + 2 \beta^2 \epsilon \rho_2 \right] \}^{-1}. \quad (6.84) \end{aligned}$$

When the equations

$$(R_2)_{CR} = (h' S_{22} + h'' S_{22})_{CR}, \quad (6.85)$$

$$D = (\hat{h} \bar{h})^2 \frac{{}'h''h'E}{h + h} , \quad (6.86)$$

$$\xi = \frac{{}'S_{22}}{''S_{22}} , \quad (6.87)$$

are introduced into the definition of K_p , i.e.,

$$K_p = \frac{(R_2)_{CR}}{P_0}$$

a rigidity parameter is obtained, namely,

$$\frac{2\pi D ({}'h + ''h)}{{}''h \left({}'h + \frac{{}''h}{\xi} \right) (1+m) L_x^2} = 2(1-\mu^2) \frac{({}'S_{22})_{CR}}{K_p} . \quad (6.88)$$

Eqs. (6.84) and (6.88) constitute the solution to the problem. For a given sandwich plate under a given state of stress in the facings, the buckling coefficient K_p can be calculated from Eq. (6.84) for various buckling modes, and the rigidity parameter can then be determined from Eq. (6.88). Comparison of the calculated rigidity parameter with the actual plate shows if buckling will occur.

As a check on the validity of the analysis, the equations will now be compared with the work of Chang, Ebcioğlu, and Baltes [26] in which the following was assumed,

1. The face layers are the same material.
2. The state of stress in the face layers is the same.
3. The middle layer is orthotropic.

4. The plate buckles in the simplest mode.
5. The face layer material is incompressible.
6. A correction factor can be applied to account for compressibility as given by Seide and Stowell [28].

From assumption #1 and Eq. (6.83),

$$m = \frac{{}'h}{''h} . \quad (6.89)$$

From assumption #2 and Eq. (6.87),

$$\xi = 1.0 . \quad (6.90)$$

From assumptions #1 and #2 together,

$${}'R_{\alpha\beta\gamma\delta} = {}''R_{\alpha\beta\gamma\delta} ;$$

and, therefore, from Eq. (6.59),

$$\omega = \frac{{}'h}{''h} . \quad (6.91)$$

From assumption #3, reference [26] has defined the parameter

$$g = \frac{\bar{G}_x}{\bar{G}_y} \quad (6.92)$$

which is the ratio of the core shear moduli in the θ_1 and θ_2 directions, respectively. For comparison, it is necessary to take

$$g = 1.0 . \quad (6.93)$$

Assumption #4 requires that

$$g = 1.0, \quad (6.94)$$

see Eqs. (6.79). Assumption #5 implies that $\mu = 1/2$. Therefore, the material constants, from Eqs. (6.76) - (6.78) become

$$K_1 = \infty, \quad G_s = \frac{1}{3} E_s, \quad G_t = \frac{1}{3} E_t. \quad (6.95)$$

Then, from Eqs. (6.69) - (6.74) and (6.83),

$$\begin{aligned} Z_{11} = Z_{33} &= 1.0, & Z_{12} &= \frac{1}{2}, \\ Z_{22} &= 1 - \frac{3}{4} \frac{G_s - G_t}{G_s}, & \epsilon &= \frac{E}{E_s} = \frac{1}{5}. \end{aligned} \quad (6.96)$$

From assumption #6, the corrected critical stress is given by the equation

$$(\dot{S}_{CR})_{\mu} = (\dot{S}_{CR})_{\mu=1/2} \frac{\frac{3}{4}}{(1-\mu^2)} \frac{(K_1)_{\mu}}{(K_1)_{\mu=1/2}} \quad (6.97)$$

where K_1 is the elastic buckling coefficient given in reference [29] as

$$\begin{aligned} K_1 = & \left\{ (1+g\beta^2) \left[(1-\mu)(1+\beta^2)^2 r + 2\beta^2(g+\beta^2) \right] \right. \\ & \left. - 2\beta^4(1-g)^2 \right\} \times \left\{ (1-\mu)(1+\beta^2)^2 r + 2\beta^2(g+\beta^2) \right\} r \\ & + \beta^2 \left[(1-\mu)(1+g\beta^2) r + 2\beta^2 g \right] \}^{-1}. \end{aligned}$$

For the special case of $g = 1.0$, the above reduces to

$$K_1 = (1 + \beta^2)^2 \left[(1 + \beta^2)r + \beta^2 \right]^{-1} \quad (6.98)$$

which is independent of Poisson's ratio. Therefore, Eq. (6.97) becomes

$$(\dot{S}_{CR})_{\mu} = \frac{\frac{3}{4}}{(1 - \mu^2)} (\dot{S}_{CR})_{\mu = \frac{1}{2}} \quad (6.99)$$

When Eqs. (6.89) - (6.91), (6.94) and (6.96) are substituted into

Eqs. (6.84) and (6.88) the result is

$$\begin{aligned} K_p = \zeta \left\{ (1 + \beta^2) \left[\frac{1}{2} (Z_{22} + 4Z_{22}\beta^2 - 2\beta^2 + \beta^4) r \zeta \right. \right. \\ \left. \left. + 2\beta^2 (Z_{22} + \beta^2) \right] - 2\beta^4 (Z_{22} - 1) \right\} \times \left\{ r \zeta \left[\frac{1}{2} (Z_{22} \right. \right. \\ \left. \left. + 4Z_{22}\beta^2 - 2\beta^2 + \beta^4) r \zeta + 2\beta^2 (Z_{22} + \beta^2) \right] \right. \\ \left. \left. + \beta^2 \left[\frac{1}{2} (1 + \beta^2) r \zeta + 2\beta^2 \right] \right\}^{-1}, \quad (6.100) \end{aligned}$$

$$\frac{2\pi^2 D}{L_x^2 (h' + h'')} = \frac{3}{2} \frac{(\dot{S}_{22})_{CR}}{K_p}, \quad (6.101)$$

which is identical with the equations of reference [26] for $g = 1.0$.

Therefore, the solutions agree.

As mentioned by Nadai [30], Poisson's ratio is variable in the transition region of elastic to plastic stress, which suggests a possible refinement for the correction formula Eq. (6.99), namely,

$$(\dot{S}_{CR})_{\mu_v} = \frac{\frac{3}{4} (\dot{S}_{CR})_{\mu = \frac{1}{2}}}{(1 - \mu_v^2)} \quad (6.102)$$

where μ_v is the variable Poisson's ratio. Nadai [30] gives an equation for the approximation of μ_v as

$$\mu_v = \mu_p - (\mu_p - \mu) \frac{\gamma_y}{\gamma_{22}}, \quad \gamma_{22} \geq \gamma_y \quad (6.103)$$

where μ_p is Poisson's ratio in the plastic range, γ_y is the elastic strain at the yield point, and γ_{22} is the total uniaxial strain.

For uniaxial stress, Eq. (3.96) can be solved to show that

$$\gamma_y = \frac{1}{E} S_y,$$

$$\gamma_{22} = \frac{1}{E_s} S_{22}, \quad \gamma_{22} \geq \gamma_y,$$

where S_y is the yield stress in compression. Then, with the usual assumption that $\mu_p = 1/2$, Eq. (6.103) becomes

$$\mu_v = \frac{1}{2} - \left(\frac{1}{2} - \mu \right) \left(\frac{E_s}{E} \right) \left(\frac{S_y}{S_{22}} \right), \quad S_{22} \geq S_y. \quad (6.104)$$

It should be realized that the expressions for K_p and Z_{11} , Z_{12} , etc., are simplified by the assumption of an incompressible face material; however, this assumption can readily be waived if one has access to an electronic computer. Therefore, it should be of interest

to compare the results obtained with and without assumptions #5 and #6. For a compressible face material, Eqs. (6.84) and (6.88) become

$$\begin{aligned}
 K_p = & \left\{ (1 + \beta^2) \left[\frac{1}{2} (Z_{22} Z_{33} + 4 Z_{11} Z_{22} \beta^2 - 2 Z_{12} Z_{33} \beta^2 \right. \right. \\
 & - 4 Z_{12}^2 \beta^2 + Z_{11} Z_{33} \beta^4) r + 2 \beta^2 \epsilon (Z_{22} + Z_{11} \beta^2) \Big] \\
 & - 2 \beta^4 \epsilon (Z_{11} + Z_{22} - Z_{33} - 2 Z_{12}) \Big\} \times \left\{ r \left[\frac{1}{2} (Z_{22} Z_{33} \right. \right. \\
 & + 4 Z_{11} Z_{22} \beta^2 - 2 Z_{12} Z_{33} \beta^2 - 4 Z_{12}^2 \beta^2 + Z_{11} Z_{33} \beta^4) r \\
 & + 2 \beta^2 \epsilon (Z_{22} + Z_{11} \beta^2) \Big] + \beta^2 \epsilon \left[\frac{1}{2} (1 + \beta^2) Z_{33} r + 2 \beta^2 \epsilon \right] \Big\}^{-1}, \\
 & \hspace{25em} (6.105)
 \end{aligned}$$

$$\frac{2\pi^2 D}{L_x^2 (h' + h'')} = 2(1 - \mu^2) \frac{(\dot{S}_{22})_{CR}}{K_p}, \quad (6.106)$$

when Eqs. (6.89) - (6.91) and (6.94) are employed. In these latter equations, the general expressions for Z_{11} , Z_{12} , etc. are to be used, i.e., Eqs. (6.69) - (6.74) and (6.83).

To obtain actual numerical results, a computer program (Fortran IV) was written. The program along with a description of its usage is given in Appendix A.

6.6. Results and Conclusions

For analysis, a square sandwich plate (actually, $\beta = 1.0$) with 17-7 PH stainless steel facings was chosen. The properties of 17-7 PH stainless steel are given in Appendix B. Under the assumption of both compressible and incompressible material, solutions were obtained for a wide range of stresses, extending well beyond the 0.02% yield point, by using the computer program given in Appendix A. The results are shown in Figs. 4 and 5.

Fig. 4 clearly shows that if the simpler, incompressible theory is used the results must be corrected. Although conservative, the correction given by Seide and Stowell [28] deviates considerably from the correct (compressible theory) solution at the higher stresses. Much better agreement is obtained by using the variable Poisson's ratio in the correction equation; unfortunately, however, this procedure produces nonconservative results for an $r > .5$, where r is the core shear stiffness parameter.

The solution for compressible theory is shown in Fig. 5. With r as a parameter, several curves are shown.

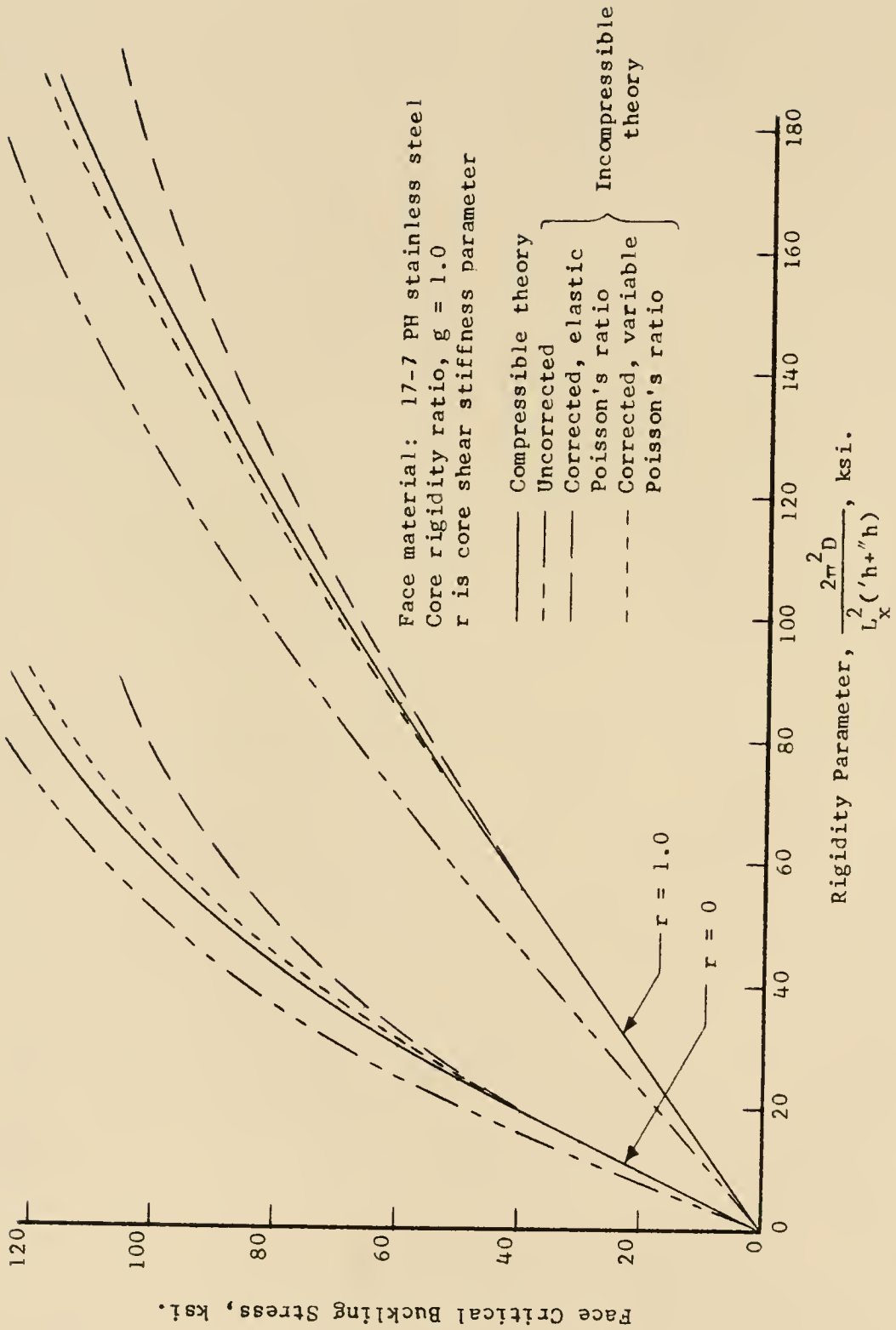


Fig. 4. Critical buckling stress, comparison of compressible and incompressible theories

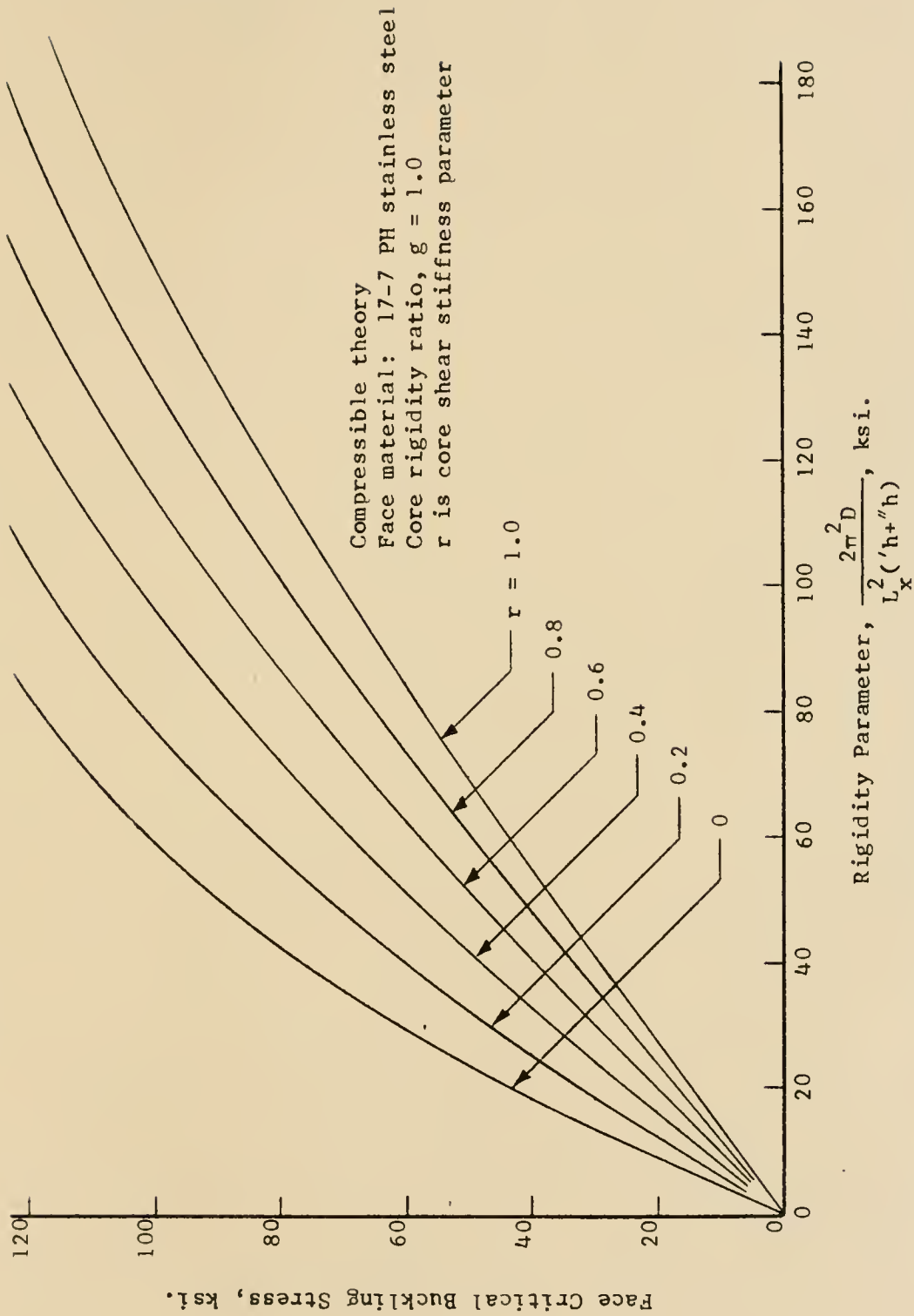


Fig. 5. Face critical buckling stress vs. rigidity parameter for square sandwich plate under uniaxial load

APPENDICES

A. COMPUTER PROGRAM

The solutions, as given by Eqs. (6.99) - (6.102) and (6.104) - (6.106) are evaluated by means of the accompanying Fortran IV computer program. The input data required are the material properties of the face layers; in particular, each data card must have the following information: compressive stress; Young's modulus; tangent modulus; secant modulus; and Poisson's ratio. The data are to be punched under the format (4E10.0, F10.0). Naturally, as many data cards can be employed as desired. With this in mind, the first card of the data deck must specify the number of following cards; and, in addition, the 0.02 per cent yield stress. The format for punching the first card is (I4, E10.0).

Besides the input data, the equations are dependent upon the parameters β and r , which are controlled by program deck cards. If an aspect ratio β other than one is desired, say four, then the card

$$B = 1.0$$

must be replaced by the card

$$B = 4.0$$

The core shear stiffness parameter r is controlled by the two cards

$$70 \ R = 0.$$

$$50 \ IF \ (R-1.1) \ 40,40,20$$

The output is completely self-explanatory except that the letter P is used to represent Poisson's ratio. The program is as follows:

```

C      PLASTIC BUCKLING
      WRITE (6, 10)
10  FORMAT (72H PLASTIC BUCKLING OF SANDWICH PLATE WITH COMPRESSIVE LO
      AD IN Y DIRECTION / 66H NUMBER OF HALF WAVES, IN X DIRECTION = 1.0
      1, IN Y DIRECTION = 1.0)
      D = 1.0
      WRITE (6, 80) D
80  FORMAT (15H ASPECT RATIO = , F3.1)
      READ (5, 65) J, SS
65  FORMAT (I4, E10.0)
      DO 20 I=1,J
      READ (5, 30) S, D, DS, DT, P
30  FORMAT (4E10.0, F10.0)
      WRITE (6, 11)
11  FORMAT (5X, 98H COMPRESSIVE STRESS YOUNG'S MODULUS SECANT MODULU
      IS TANGENT MODULUS POISSON'S RATIO (VARIABLE) )
      IF (S-SS) 23, 23, 24
22  Z=.5-((.5-P)*SS*DS)/(S*D)
      GO TO 21
23  Z=P
21  WRITE (6, 12) S, D, DS, DT, P, Z
12  FORMAT ( 9X, E10.4, 4X, 3(6X, E10.4), 2E15.4)
      WRITE (6, 60)

```

```

60 FORMAT (10X,14H CORE RIGIDITY,8X,22H BUCKLING COEFFICIENTS, 10X,
143H RIGIDITY PARAMETER CORRECTED STRESS / 13X, 96H PARAMETE
1R COMPRESSIBLE INCOMPRESSIBLE ELASTIC COMPRESSIBLE INCOMPRESSIBLE
1 CONST. P VARIABLE P )

G=D/(2.*(1.+P))

GS=DS/(3.-(1.-2.*P)*DS/E)

GT=DT/(3.-(1.-2.*P)*DT/U)

XK=D/(3.*(1.-2.*P))

XN=3.*XK

XH=(4.*GS*(XN+GS))/(XN+4.*GS)

YG=(2.*GS*(XN-2.*GS))/(XN+4.*GS)

S1=.4714*S

XR=(3.*(GT-GS))/(S*S*(XN+4.*GS)*(XN+3.*GS+GT))

A11=(2.*GS+YG+XR*(2.*GS*G)**2)/XH

A22=(2.*GS+YG+XR*((XN+2.*GS)*S)**2)/XH

A33=4.*GS/XH

A12=(YG-XR*2.*GS*(XN+2.*GS)*S*S)/XH

B22=1.-.75*((DS-DT)/DS)

PH=DS/D

XE=D/((1.-P*P)*XH)

A1=.5*(A22*A33+4.*A11*A22*B*B-2.*A33*A12*B*B-4.*A12*A12*B*B+A11*A3
13*B**4)

A2=2.*B*B*XL*(A22+A11*B*B)

```

```

A3=2.*XE*B**4*(A11+A22-A33-2.*A12)
A4=.5*B*B*XE*(1.+B*B)*A33
A5=2.*(D*B*XE)**2
B1=.5*(D22+4.*B22*B*D-2.*E*B+D**4)*PH
B2=2.*E*D*(D22+3*B)
B3=2.*(B22-1.)*B**4
B4=.5*D*D*(1.+B*B)*PH
B5=2.*B**4
C1=(1.+B*B)**2
C2=(1.+D*E)
PA=2.*S*(1.-P*P)

70 R=0.

50 IF (R-1.1) 40,40,20

40 XKP1=(1.+E*D)*(A1*R+A2)-A3
XKP2=R*(A1*R+A2)+A4*R+A5
XKP=XKP1/XKP2

YKP1=(1.+D*D)*(D1*R+B2)-B3
YKP2=R*PH*(D1*R+D2)+B4*R+B5
YKP=(PH*YKP1)/YKP2

YK1=C1/(R*C2+2*D)

ZN=.75/(1.-P*P)

Z1=PA/XKP
Z3=1.5*S/YKP

```

Z2=ZN*S

ZM=.75/(1.-Z*Z)

Z4=ZM*S

WRITE (6,61) R, XKP, YKP, YK1, Z1, Z3, Z2, Z4

61 FORMAT (15X, F6.2, 2X, 3F12.5, 2X, 2E11.3, 4X, 2E11.3)

R=R+.2

GO TO 50

20 CONTINUE

RETURN

END

B. MATERIAL PROPERTIES

Here, the material properties are given for 17-7 PH stainless steel. The stress-strain curve was taken from the book by Watter and Lincoln [31] and is plotted in Fig. 6. From the curve, the various moduli (i.e., Young's, secant, and tangent) were measured. Young's modulus was found to be 28.6×10^6 psi, and the secant and tangent moduli are plotted in Fig. 7. Also, the 0.02% yield stress was found to be 68,000 psi. In addition, from reference [32], Poisson's ratio was taken to be 0.28.

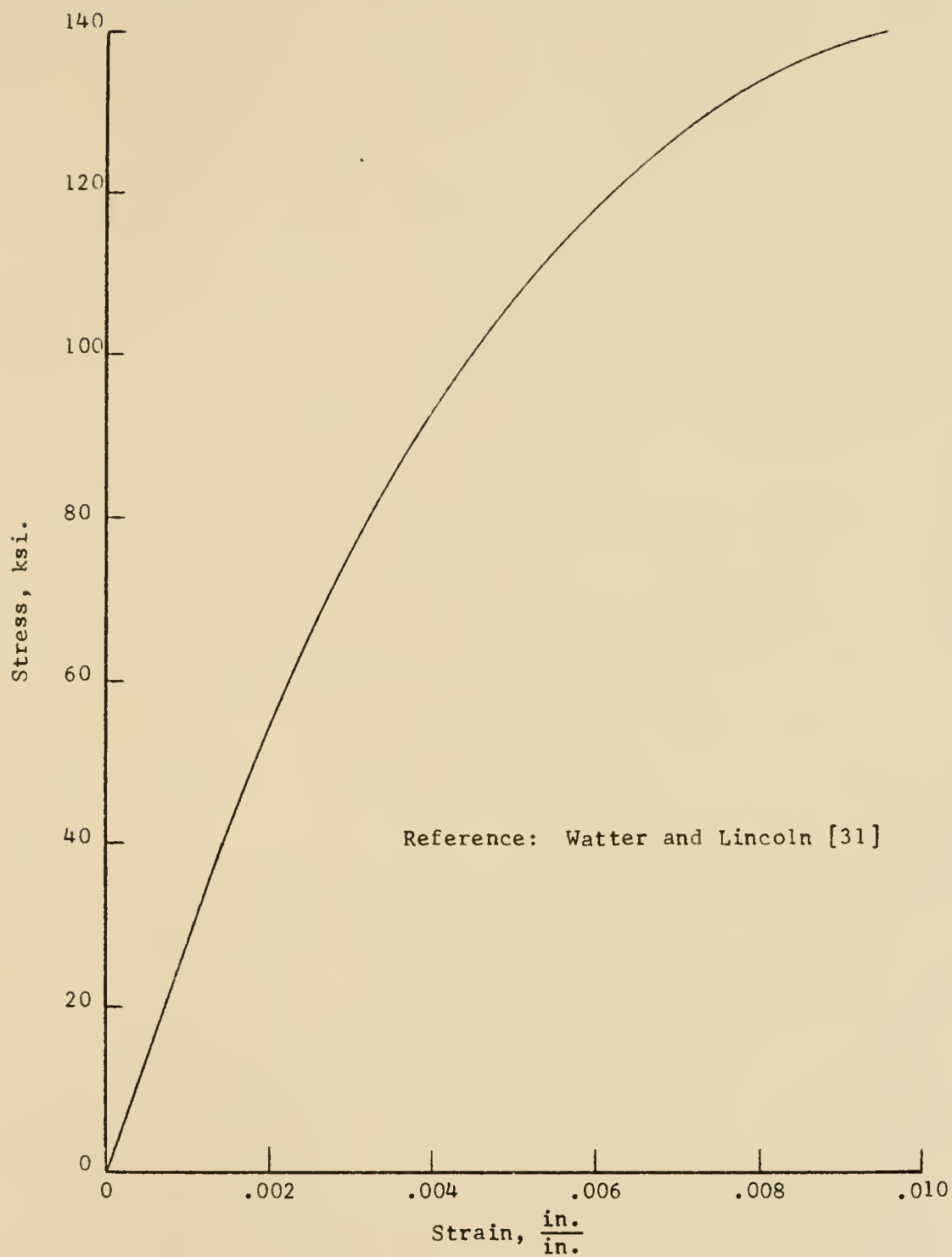


Fig. 6. Stress-strain curve for 17-7 PH stainless steel .
in compression

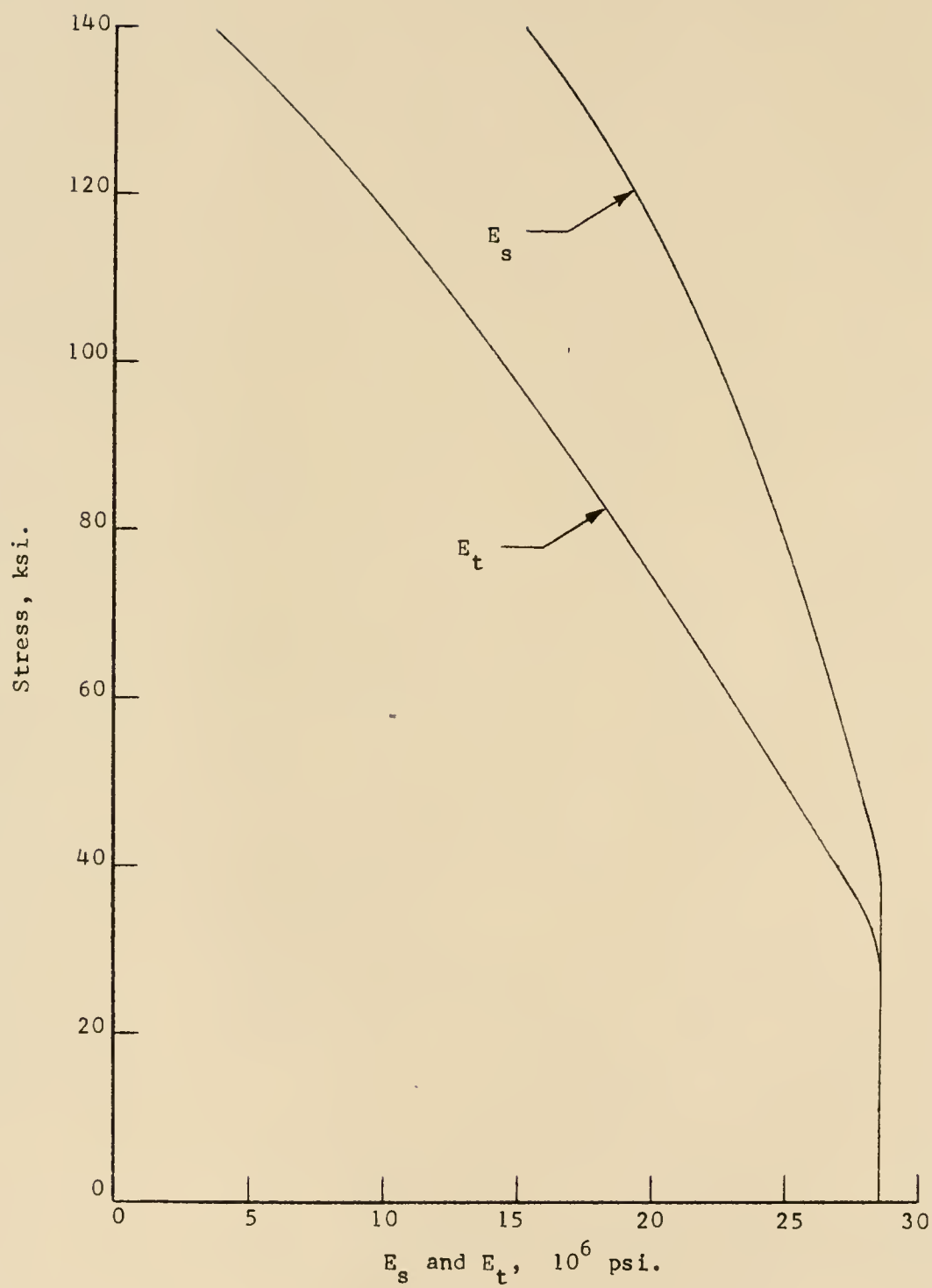


Fig. 7. Tangent and secant moduli for 17-7 PH stainless steel

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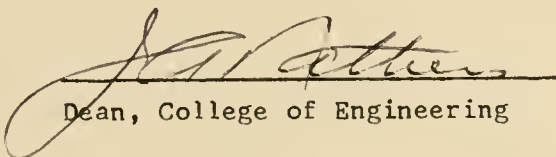
BIOGRAPHICAL SKETCH

Fred Jacob Monzel was born January 13, 1931, at Grand Rapids, Michigan; but most of his childhood was spent in Indianapolis, Indiana, where he was graduated from Arsenal Technical High Schools in June, 1948. Fred worked as a clerk for the Texas Oil Company from 1948 to 1951, and then he enrolled at Purdue University. In February, 1955, he received the degree of Bachelor of Science in Mechanical Engineering with distinction from Purdue University. While at Purdue, Fred was a member of Pi Tau Sigma (mechanical engineering honorary) and Tau Beta Pi (all engineering honorary); also, he was made an honorary member of ASTM. Fred was then employed as a stress analyst by the Allison Division of General Motors Corporation until July, 1956. From 1956 until 1962, Fred was a technical engineer with the Flight Propulsion Division of the General Electric Company, engaged in the mechanical design of jet engines. Then, in February, 1962, he accepted the position of senior engineer with the Astronuclear Laboratory of Westinghouse Electric Corporation. While at Westinghouse, Fred was selected for a special one-year assignment which sent him to Los Alamos Scientific Laboratories to provide technical assistance on the design and development of nuclear rocket engines. In September, 1963, Fred took a leave of absence from Westinghouse to enroll in the Graduate School of the University of Florida. The University of Florida awarded him with a NASA traineeship and graduated him with the degree of

Master of Science in Engineering in December, 1964; since then, Fred has been studying in the Department of Engineering Mechanics at the University of Florida in the pursuit of a doctorate degree.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April, 1967

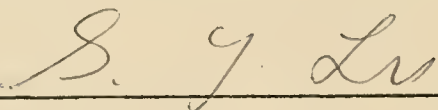

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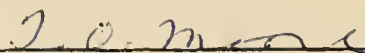
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